

# Dynamics of non-metric manifolds

Alexandre Gabard and David Gauld\*

January 13, 2013

*Je ne crois donc pas avoir fait une œuvre inutile en écrivant le présent Mémoire; je regrette seulement qu'il soit trop long; mais quand j'ai voulu me restreindre; je suis tombé dans l'obscurité; j'ai préféré passer pour un un peu bavard. (Henri Poincaré, 1895, introducing his *Analysis situs*.)*

ABSTRACT. An attempt is made to extend some of the basic paradigms of dynamics— from the viewpoint of (continuous) flows—to non-metric manifolds.

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## 1 Introduction

The present paper, by far not having the intrinsic *charism* of Poincaré's *Mémoire*, may share some of the supposed discursive defects—albeit in the more annoying way that our loquaciousness, instead of reflecting a wealth of new insights, resulted rather from a poor understanding of a somewhat exotic subject-matter. At any rate, like Poincaré, we shall put ourselves at the cross-intersection of the two paradigms “manifolds” and

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\*Supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand.

“dynamics”. Albeit extensively studied since and under his impulsion (1880–1913), the subject still contains certain *places réputées jusqu’ici inabordables*<sup>1</sup>, even in seemingly anodyne situations. Elusive, open problems belonging to the genre are: *Hilbert’s 16th problem* for the number and mutual disposition of *cycles limites* of polynomial vector fields in the plane, or the question as to whether the 3-space  $\mathbb{R}^3$  (or the 3-sphere) admits a flow with *all* orbits dense (*Gottschalk conjecture* formulated in 1958 [27], along the tradition of Poincaré–Hadamard–Birkhoff–Morse–Hedlund, not to mention Markoff [46], and reposed by Smale on many occasions).

Beside such difficult questions, a sizeable portion of theory is well buoyed, forming so to speak a main-stream of knowledge. Our task will merely reduce to selecting among such well-oiled mechanisms, those capable of a *tele-transportation* beyond the metric realm. To gain some swing, we shall briefly loop-back at some early history, aiding—at least fictionally—to circumvent better the nature of the “main-stream” in question.

During the 19th century the concept of space enjoyed a golden reconfiguration producing the fruitful concept of *manifolds* (Gauss, Lobatschevsky, Riemann, etc). This involves the idea of a space locally modelled over some “flat” number-space like  $\mathbb{R}^n$ . Gradually the “manifold” idea came to its clear-cut precision but perhaps only through specialisation of the much broader concept of a *topological space* (Hilbert 1902, Fréchet 1906, Hausdorff 1914). Among the earliest axiomatisation of manifolds we count: Weyl 1913 [70] (with a triangulated influence of Brouwer), Kerékjártó 1923 [40, p. 5] for pure *topological manifolds* ( $=C^0$ -manifolds), Veblen–Whitehead 1931 [68] for differential manifolds.

Beside this purely “spatial” development, physics (typically Newtonian mechanics) set forth the description of natural phenomena evolving in time via differential equations. The associated flows became Poincaré’s fleuron to launch the great qualitative programme (stability, instability, chaos, etc). Eventually, an easy abstraction allows one to think about *flows* without reference to the differential calculus, as a topological group action of the real line  $\mathbb{R}$  over a certain topological space  $f: \mathbb{R} \times X \rightarrow X$ . (This shift of viewpoint occurs by Kerékjártó 1925 [41], Markoff 1931 [46], and in full virtuosity by Whitney 1933 [71].)

Both notions “manifolds” and “dynamics” turned out to be quickly intermingled in a “space-time” companionship. E.g., as early as 1839, Gauss in his *Allgemeine Theorie des Erdmagnetismus* [26, Artikel 12, p. 134–135], noticed that the speculation that the earth might not have a unique magnetic north pole would ineluctably create some other “hybrid” pole which is neither a north nor a south pole<sup>2</sup>. This can of course be recognised as an early form of the *Poincaré–Hopf index theorem* (and the related hairy-balls theorems). An intermediate link from Gauss to Poincaré is the Kronecker index (1869) which allowed many of the forerunner, e.g. Bohl 1904 [12] to anticipate by some years some of the contributions of Brouwer. The Poincaré–Hopf index theorem (and the allied Lefschetz fixed point theory) appear as prominent outcome of this era, altogether incarnating one of the most basic link between the shape of a space and its dynamics (thus, a good candidate to keep in mind for tele-transportation). Without rushing on this, recall also the consequence that *a closed manifold accepts a non-stationary flow if and only if its Euler characteristic vanishes* (Hopf [35], [1]). [In passing, the reverse implication does *not* seem to have been firmly established for  $C^0$ -manifolds (more on this in Section 5.9). The direct sense follows, of course, from Lefschetz’s extension 1937 [43] of his theory to the class of compact metric ANR’s.]

A “general” manifold—defined merely via the locally Euclidean desideratum—because of its naked elegance is capable of various forms of perversities, which are traditionally brought into more respectableness through additional restrictions: e.g., the Hausdorff separation axiom, metrisability of the topology, compactness, differential structures, Riemannian metrics, etc.

The modest philosophy of our text is that while the specialisation to metric manifolds is essential for “quantitative” problems (e.g., the classification of 2-manifolds<sup>3</sup>) there is some respectable “qualitative” principles which are sufficiently robust to hold non-metrically. Examples of this vein are the Jordan separation theory, the Schoenflies theorem (any circle bounds a disc). Thus, the Poincaré–Bendixson theory—relying on the *sack argument* acting as a trap for trajectories on a surface where Jordan separation holds true—also propagates non-metrically. From it and Schoenflies, one can draw a *hairy ball theorem* for  $\omega$ -bounded<sup>4</sup> simply-connected surfaces, yielding a wide extension of the fact that the 2-sphere cannot be brushed. (By a *brush*, we shall mean

<sup>1</sup>To quote again—this time loosely—Poincaré [60, p. 82].

<sup>2</sup>To quote Gauss more accurately (*loc. cit.*): “Von einigen Physikern ist die Meinung aufgestellt, dass die Erde zwei magnetische Nordpole und zwei Südpole habe: [...]—Sehen wir von der wirklichen Beschaffenheit der Erde ab, und fassen die Frage allgemein auf, so können allerdings mehr als zwei magnetische Pole existiren: es scheint aber noch nicht bemerkt zu sein, dass sobald z. B. zwei Nordpole vorhanden sind, es nothwendig zwischen ihnen noch einen dritten Punkt geben muss, der gleichfalls ein magnetischer Pol, aber eigentlich weder ein Nordpol noch ein Südpol, oder, wenn man lieber will, beides zugleich ist.” This, and other early history, are surveyed in the famous paper of Dyck [21]. Recall also the rôle of Listing, both for its link with Maxwell and as a forerunner of “homology”.

<sup>3</sup>Worked out by Möbius 1863 [48], Jordan, Klein, Weichold 1883, Dyck 1888 [21] Dehn–Heegaard 1907, etc. and in the non-compact case Kerékjártó 1923 [40].

<sup>4</sup>Recall that a  $\omega$ -bounded space is one such that any countable subset admits a compact closure. This point-set concept when particularised to manifolds allows one to hope recovering some of the “finistic” virtues of compact manifolds beyond the metric realm, cf. e.g. Nyikos’s bagpipe theorem to be discussed below.

a flow without stationary points—following the terminology of Beck [6].)

Beside this pleasant propagation of certain robust paradigms, it must be confessed that some other fails dramatically. A typical disruption of this kind occurs to *Whitney's flows*. First, the natural desideratum of attaching to a brush its induced foliation works universally in class  $C^1$  and topologically in low-dimensions  $\leq 3$  (Whitney 1938 [72]), but not in higher-dimensions in view of wild  $C^0$ -actions à la Bing (cf. Chewning 1974 [19]). Next, this process admits a reverse engineering, which *creates* a flow-motion compatible with a given one-dimensional orientable foliation (Whitney 1933 [71]). Thinking of such *Whitney's flows*—in rough caricature—as obtained by parameterizing leaves by arc length indicates a definite *metric* sensitivity. It is not surprising therefore, that one can easily experiment non-metric failures (Propositions 2.6 and 3.4) even when all leaves are *short* (i.e., metric). In the same vein, Hopf's issue that a vanishing Euler characteristic is sufficient for a brush lacks a non-metric counterpart (Remark 4.12 discusses the example of the connected sum of two long planes  $\mathbb{L}^2$ ).

Accordingly, results from the classical theory can be sorted out under the following three headings depending on their ballistic when catapulted outside the metric stratosphere:

(1) *Stable theories and theorems* (“*passe-partout*” in Grothendieck's jargon): those sufficiently robust as to hold non-metrically. Examples: Jordan, Schoenflies, Poincaré-Bendixson, two-dimensional hairy-ball theorems, *phagocytosis*, i.e. the aptitude for a cell (chart) to engulf any countable subset of a manifold, cf. Gauld [25]<sup>5</sup>. This consequence of Morton Brown's monotone-cell-union theorem, also turns out to have multiple dynamical repercussions, as we shall see.

(2) *Unstable theorems and vacuous paradigms*: theorems breaking down outside the metric world (Example: Whitney's flows, Hopf's brushes when  $\chi = 0$ , Beck's technique for slowing-down flow lines); and paradigms which do not survive by lacking any single non-metric representative: Lie group structures, minimal flows, global parallelism (à la Stiefel). [Of course we do *not* claim that those theorems are less good than those of the first category (1), but rather that their non-metric collapses adumbrate a deeper geometric substance.]—And finally:

(3) *Chaotical (undecided) paradigms*: principles which as yet (under our fingers) could not be ranked into one of the previous two headings. Examples: Finiteness property for the singular homology of  $\omega$ -bounded manifolds and Lefschetz fixed point theorem, hairy-ball theorems for  $\omega$ -bounded manifolds with  $\chi \neq 0$ , existence of smooth structures in low-dimensions  $\leq 3$  (Spivak-Nyikos question, ref. as in [23]), existence of transitive flows on separable manifolds of high-dimensions  $\geq 3$  (à la Oxtoby-Ulam [57], Sidorov [65], Anosov-Katok [2]).

For simplicity, we count as a subclass of (3) *truly chaotical (undecidable)* results which are known to be sensitive on some axiomatic beyond ZFC (Zermelo-Fraenkel-Choice). Example: *perfect normality*, i.e. the possibility of cutting-out an arbitrary closed set as the zero-locus of a real-valued continuous function. [Work of M. E. Rudin, Zenor.]

The by-standing synoptic diagram may help as a navigation system; it shows:

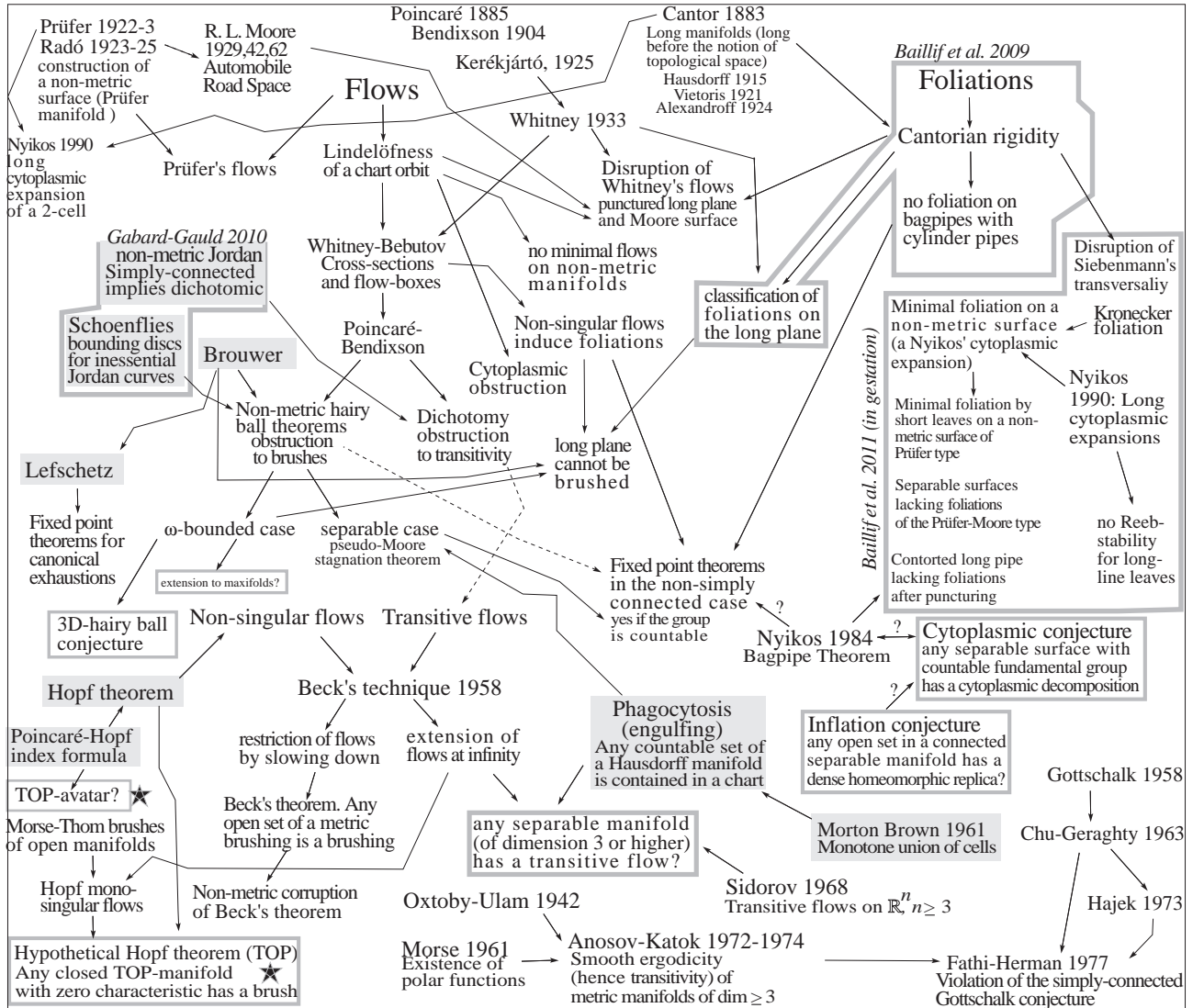
- Two islands delineated by frames, packaging results of previous papers by the authors. Since the present paper emphasises the viewpoint of flows, the paper [4] (concerned with foliations) is not an absolute prerequisite. The note [23] (Jordan and Schoenflies in non-metrical analysis situs) will be used in some arguments.
- Shaded regions mark classical theorems that might be adumbrative of certain non-metric prolongations. Admittedly, certain aspects of the non-metric theory of manifolds is just a matter of transposing classical results via transfinite repetition or by Lindelöf approximation of “small” metric sub-objects. (The paper [23] certainly provides a good illustration of this reductionism.)
- Framed rectangles correspond to conjectures delineating severe limitations in the authors' knowledge. The two starred frames correspond to purely metric questions, as to whether the Poincaré-Hopf index formula, eventually also the Hopf existence theorem for brushes when  $\chi = 0$ , generalise to  $C^0$ -manifolds (cf. Section 5.9 for some heuristics).

## 1.1 Non-metric manifolds: a short historiography

Perhaps first, some few words looping back to the sources of non-metric manifold theory. The top of the iceberg emerged in Cantor's 1883 *Punktmannichfaltigkeiten* [16, p. 552], where the *long ray* and allied *long line* were suggested. A second generation of natural—indeed perfectly geometric—examples occurred to Prüfer and Radó 1922/1925 [62], [63], as a byproduct of their investigations of Weyl's treatment of *Die Idee der Riemannschen Fläche* [70]. A similar vision occurred, seven years later in 1929, to R. L. Moore in the form of an *Automobile Road Space* (see the report by F. Burton Jones [39]<sup>6</sup>), thereby rediscovering—apparently

<sup>5</sup>In the compact case such phagocytosis appears in the work of Morton Brown [13], and Doyle-Hocking.

<sup>6</sup>Quoting from F. Burton Jones [39]: “Coming home from the “Boulder meeting” in the summer of 1929, Moore discovered his *Automobile Road Space*. [It] is an example of a nonseparable complete Moore space which is a 2-manifold.” This fancy name corresponds to what is nowadays commonly termed the *Prüfer surface* or the *Prüfer manifold* (cf. Radó 1925 [63], Carathéodory 1950 [18], Nevanlinna 1953 [53], Calabi-Rosenlicht 1953 [15], Ganea 1954 [24]).



independently—the example of Prüfer. Moore’s contribution (published only in 1942 [49]) is a certain “twist” in the Prüfer construction producing separable simply-connected examples. A noteworthy feature of both the Prüfer and Moore constructions is that they are not isolated specimens, but rather more “fabrics” engendering a variety of civilised, easy-to-visualise examples. From the set-theoretical viewpoint, the real eclosion of the subject—yet another heritage of R. L. Moore’s School<sup>7</sup>—is incarnated by the contributions of M. E. Rudin, Zenor. The 1984 paper of Nyikos [55] is the best initiation to the vertiginous depth of the non-metric universe (even in the 2-dimensional, simply-connected setting). It also achieves a subtle balance of point-set versus combinatorial methods, culminating to the *bagpipe theorem*, showing that the subclass of  $\omega$ -bounded surfaces behaves like the familiar compact 2-manifolds, save for the presence of *long pipes* emanating out from the *bag*, while travelling at such sidereal distances as to violate any metrisation. Those pipes could be thought of as circle-bundles over a closed long ray, yet their real structure is in general somewhat more mysterious. In particular they do not necessarily admit a *canonical* exhaustion by compact bordered cylinders. This plague of “wild pipes” will cause us some troubles, when attempting to tele-transport the Lefschetz fixed point theory.

For more intelligibility and to the convenience of the (non-specialised) reader, let us recall that the *bordered*<sup>8</sup> Prüfer surface,  $P$ , can be thought of as the open upper-half plane  $H = \mathbb{R} \times \mathbb{R}_{>0}$  plus some ideal points materialised by rays rooted on the horizontal boundary line  $\{y = 0\}$  and pointing into  $H$ . All this data can be naturally topologised, to produce a certain *bordered* surface,  $P$ , whose interior is an open 2-cell and whose boundary splits into a “continuum”  $\mathfrak{c} = \text{card}\mathbb{R}$  of components each homeomorphic to the real line. Thus, faithful

<sup>7</sup>From the dynamical viewpoint we already alluded to R. H. Bing’s impact—via wild topology—on a fundamental question of Whitney in 1933 (on cellular cross-sections), implying a radical divorce (already in the metrical realm, of course) between the topological and smooth approach to dynamical systems. A similar divorce occurred earlier with (discrete) transformation groups, (recall Bing’s involution of the 3-sphere (1952) [10]). More on this in Section 2.3.

<sup>8</sup>Following Ahlfors-Sario, we employ *bordered manifold* as a synonym of “manifold-with-boundary”, which seems to us better than the “bounded manifolds”, used e.g. by J. H. C. Whitehead—avoiding any conflict with “ $\omega$ -bounded”.

to the automobile jargon, this Prüfer surface,  $P$ , resembles a windscreen with a continuum of wiper (not just two) each rooted at the bottom of the screen. With this picture, it is easy to visualise a non-singular flow on  $P$ , akin to a windscreen wiper motion (cf. Figure 1, left hand-side). Besides, the *Moore surface* is just the quotient of the bordered Prüfer surface,  $P$ , by gluing each of its boundary components via the identification  $x \sim -x$  (compare Figure 1, right hand-side). The figure also shows two other 2-manifolds naturally deducible from the bordered  $P$ , namely a collared version  $P_{\text{collar}} = P \cup (\partial P \times [0, \infty))$  (which turns out to be the same as the original Prüfer surface described in Radó 1925 [63]), plus its *double*  $2P = P \cup P$ , i.e., the gluing of  $P$  with a replica of itself (compare Calabi-Rosenlicht 1953 [15]).

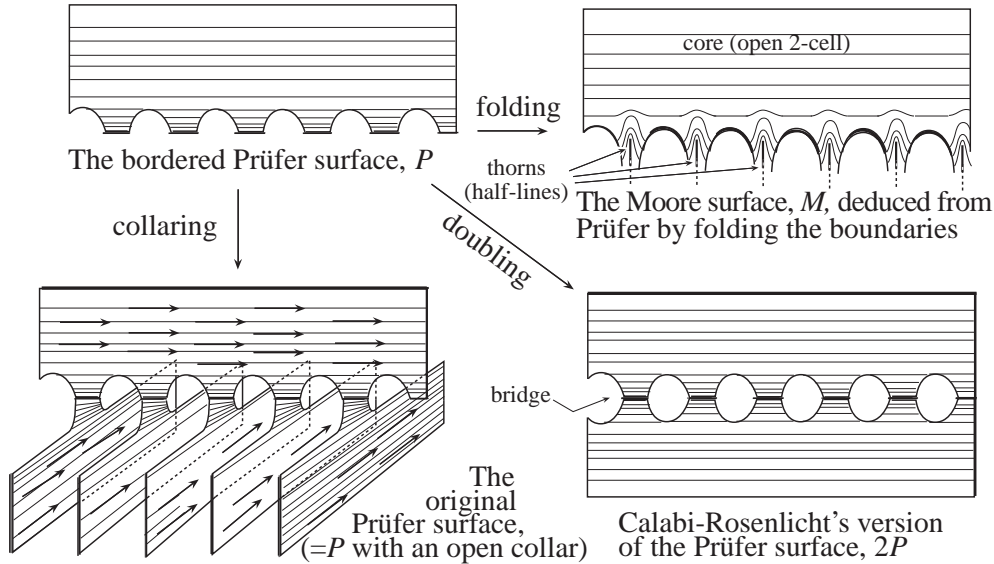


Figure 1: Artist views of the Prüfer and Moore surfaces (with galvanic currents)

## 1.2 Dynamics of flows: overview of results

The issue—that many paradigms of dynamics holds true non-metrically—has a very simple origin, rooted in the “shortness” of time, modelled by the real line  $\mathbb{R}$  (Dedekind-Cantor continuum). It implies, the orbit  $f(\mathbb{R} \times U)$  of any chart  $U$  under a flow to be Lindelöf, hence metric (Urysohn). Inside this metric flow-invariant subspace, one can draw *cross-sections* and *flow-boxes* (Whitney-Bebutov theory), yielding a straightening of the motion in the vicinity of any non-singular point. One can then ape the classical Poincaré-Bendixson theory, and establish fixed-point theorems for flows (non-metric hairy ball theorems) under weak point-set assumptions (like  $\omega$ -boundedness, and later separability), plus simple-connectivity.

Our broad tolerance for *non-metric* manifolds prompts the question of why we are not playing with longer groups, e.g., the long line  $\mathbb{L}$  as a model of time. Arguably, any such model, to deserve really the name, should at least carry the structure of a topological group. In this respect, a theorem of Garrett Birkhoff and Kakutani (1936) [9] says that *a first countable topological group is metrisable*; impeding non-metric manifolds entering the arena of topological (*a fortiori* Lie) groups. Thus, dynamics may allow big spaces but is inherently limited to short times. (Here, foliations are more flexible, as leaves can easily stretch into longness.)

Maybe the almost subconscious appeal of (continuous) flows relies in part—beside the Kriegspiel or varied physico-chemical interpretations—on the tautological observation that the real line  $\mathbb{R}$  is the building brick of *any* manifold theory (whether metric or not); a flow offering thereby an introspection of the manifold via its architectonic constituent.

Special attention was given—somewhat parallel to the interest aroused by the ergodic hypothesis—when much of the space is explored by starting from a *definite* resp. *any* initial position. This leads to the classical notions of *transitive*, resp. *minimal* flows as those having at least *one* (resp. *all*) orbits dense in phase-space. The paradigmatic climax of minimal flows, easily generated on tori (Kronecker), still leads to deep questions like the *Gottschalk conjecture* [27], already mentioned. (Assume a somniferous Riemannian oracle saying that closed *positively curved* manifolds lack a minimal flow; then beside implying Gottschalk in *all* dimensions, the result of Fathi-Herman (1977) would imply that products of odd-spheres, e.g.  $\mathbb{S}^3 \times \mathbb{S}^3$ , lack positive curvature, cracking partially H. Hopf’s puzzle from the 1930’s.) Surprisingly, the Gottschalk inquiry takes a much simpler *tournure* in the non-metric world: the Lindelöfness of any chart-orbit  $f(\mathbb{R} \times U)$  *rules out* the existence of any minimal flow on *all* non-metric manifolds (in a single stroke!). This fits into the picture that non-metric manifolds cannot

concentrate too rich structures: no group structures compatible with their topology (Birkhoff-Kakutani), no minimal systems, nor for instance a global *Fernparallelismus* in the sense of Stiefel<sup>9</sup>.

Alas, all these limitations do not close the subject, as despite not having all the symmetry perfection of the metric theory, we shall attempt to argue that there is still a rich and easy-to-experiment “geometry” of flows on non-metric manifolds<sup>10</sup>. Since minimal flows are too much demanded, we switch attention to the weaker notions of non-singular flows (alias *brushes*) resp. transitive flows and ask: *Which (non-metric) manifolds supports them?*

Answers can be obtained by a mixture of combinatorial and point-set methods. Clearly, a transitive manifold deserves to be separable (rational-times of a dense orbit); yet not necessarily metric (Example 3.6 considers a Kronecker flow on the torus, suitably Prüferised along a portion of orbit).

For metric surfaces, the standard obstruction to transitivity is *dichotomy* (i.e., any embedded circle disconnects the surface). This is the classical inference of Jordan separation on Poincaré-Bendixson, which remains activated non-metrically (Lemma 5.2). The stronger Schoenflies property, to the effect that any circle bounds a disc, acts as an obstruction to brushes, provided the surface is  $\omega$ -bounded (Theorem 4.5).

Another noteworthy obstruction to brushes involves merely general topology (viz. Lindelöfness): assume the flow  $f$  on  $M$  admits a “small” (viz. Lindelöf) *propagator* (i.e., a subset  $\Sigma \subset M$  such that the restricted flow-map  $f: \mathbb{R} \times \Sigma \rightarrow M$  is surjective) then the image  $M$  is Lindelöf.

This simple fact identifies many surfaces (e.g., the Moore surface) lacking any brush; and more generally those  $n$ -manifolds admitting a *cytoplasmic* (or *core-thorn*) *decomposition*, as defined in Section 3.4. The easy argument is best visualised on the Moore surface, whose very specific morphology—consisting of a “core” (the open half-plane of Prüfer), into which many thin “thorns” (semi-lines) are sticking in (compare Figure 1)—implies readily the core to be a propagator (under any brush). In contradistinction, when this small propagator obstruction is vacuous, we are frequently able to construct brushes on non-metric surfaces typically those of Prüfer type (Proposition 3.5); corroborating thereby the aforementioned intuition of the windscreen wiper motion (depicted on Figure 1).

The inaptitude of the Moore surface to “brush”, leads to the question, if a surface sharing *abstractly* its most distinctive topological traits (namely simply-connectedness, separability, non-metrisability and boundaryless) can support a brush. Theorem 4.18 proposes a negative answer, positively interpretable as a hairy ball theorem for this class of *pseudo-Moore* surfaces. Like the  $\omega$ -bounded hairy ball, this dual separable version derives from the same ingredients (non-metric Schoenflies, Brouwer, and Poincaré-Bendixson), plus an extra-quick owing to the *phagocytosis principle* (à la Morton Brown).

A certain “duality” seems to relate the paradigms of “ $\omega$ -boundedness” and “separability”, at different levels. First, the conjunction of both properties forces compactness. This is why, non-metrically, they represent two *totally* disjoint streams of forces. Second, the analogy goes further than the common hairy-ball theorem, for  $\omega$ -boundedness is crystallized—not to say immortalized (at least in 2-dimensions)—into the bag-pipe decomposition of Nyikos, while it seems rather likely that separability relates to what we just called cytoplasmic decompositions. Yet, this is not completely true as exemplified by the separable doubled Prüfer surface,  $2P$ , which lacks a cytoplasmic decomposition (because it has a brush). Thus, separability alone is not enough to have a cytoplasm, but restricting the fundamental group to be trivial (or even countable) might be sufficient? Such a cytoplasmic structure theory might represent a certain interest, yet we shall not address this question further. Its dynamical consequence would be that *non-metric separable surfaces with countable fundamental groups lack a brush*. Albeit, derived via a blatantly hypothetical route, this turns out to be a trivial consequence of the separable hairy ball theorem (Corollary 4.19). Since we slightly deviated in the topological register, it seems also opportune to notice that  $\omega$ -boundedness, indeed the weaker sequential-compactness, implies a form of maximality (akin to the one of closed manifolds), effecting that such manifolds are *inextensible* (or *maxifolds*); i.e., they cannot be embedded in a larger connected manifold of the same dimensionality. (This follows at once—via a clopen argument—from the invariance of the domain, which ensures openness of the image.)

Summing up, one sees—especially in two-dimensions—that the basic paradigms of dynamics (Poincaré-Bendixson, Brouwer’s fixed-point theorem) plus the idea of small propagation permit a fairly accurate answer to the question of which manifolds admits a brush resp. a transitive flow. The slightly “botanical” Section 5.3 is adumbrative of the exhaustiveness of the theoretical obstructions listed so far, by constructing surfaces with prescribed topology and dynamics. Next, what about higher dimensions?

Here our results get more fragmentary, yet some positive things happen. For instance one is tempted to transplant the Euler obstruction  $\chi \neq 0$  to the existence of brushes from the compact to the non-metric realm.

<sup>9</sup>Recall the argument of J. A. Morrow [51]: a trivialisation of the tangent bundle allows one to introduce a Riemannian metric which in turn will metricise the manifold topology, in the large.

<sup>10</sup>Of course, the issue that some non-metric manifolds are also capable of a rich “geometry” is by no mean a new age philosophy; recall Calabi-Rosenlicht’s solution [15] of Bochner’s conjecture (existence of complex-analytic structures on certain non-metric manifolds of the Prüfer type; question also implicit in Carathéodory [17, p.94].)

This can be achieved via the Lefschetz fixed-point theorem, provided some control is put on the growing mode of the manifold via so-called *canonical exhaustions* (Proposition 4.8). Yet the full punch would be a truly non-metric Lefschetz theory for  $\omega$ -bounded manifolds materialised by the following optimistic conjectures: —(1) The singular homology of such a manifold is finitely generated. —(2) The non-vanishing of the Lefschetz number of a map is a sufficient condition for the existence of a fixed point. Recall that Jaworowski 1971 [36] proves a Lefschetz fixed point theorem for any (metric or not) manifolds, but only for *compact maps*, i.e. those with relatively compact image.

Finally, we shall briefly address the topic of transitive flows. By their very definition, manifolds are locally Euclidean, allowing one to stretch about any point an open set homeomorphic to the number-space,  $\mathbb{R}^n$ . In many cases (spheres, tori, etc.) such a chart may be inflated until to cover a sizeable (indeed dense) portion of the manifold. The *phagocytosis principle*—to the effect that every countable subset of a manifold, whether metric or not, is contained in a chart—shows this to be a general feature of separable manifolds (also when non-metric, e.g., the doubled Prüfer  $2P$ ). In view of this, and the technique of Beck (allowing one to “extend”, after a suitable time-change, a flow given on a small space to a larger one), one might hope to construct transitive flows on (m)any separable manifolds of dimension  $\geq 3$ . Unfortunately, we failed to reach serious conclusions in that direction, either by removing the parenthetical “(m)” of “many” to make it an “any” (or by locating a counterexample). In other words, the well-known issue—that in dimensions  $\geq 3$  *all* metric manifolds are transitive (Oxtoby-Ulam, Sidorov, Anosov-Katok)—remains undecided for non-metric (separable) manifolds.

In conclusion, it seems that non-metric manifolds split into two types of populations, *civilised* (metric-like) against wild *barbarians*. The plague of wild pipes impeded us to formulate a universal  $\omega$ -bounded Lefschetz theory, and some separable manifolds with a wild topology at infinity (outside a phagocytosing dense chart) might troubleshoot the Beck technique. Of course, in every-day practice one mostly interacts with civilised examples (of the Cantor, Prüfer or Moore type), yet the barbarians exist—as reported by some advanced sentinels (e.g. Nyikos [55])—and potentially causes troubles to a naive-minded propagation of paradigms like those of Lefschetz or Beck. The suspense is *intact*! (As a very vague guess, the duality discussed above might suggest that barbarians are equi-distributed in both classes  $\omega$ -bounded vs. separable, so that a failure by Lefschetz would imply a failure by Beck, and vice versa?)

## 2 Flows versus foliations

Most of this Section 2 is a survey of metric results, with straightforward non-metric extensions afforded by the chart orbit trick (viz. its Lindelöfness). Thus, the reader not primarily interested in foliations, but merely in flows (easier to define, albeit of a wilder transverse nature) can skip it, and move forward to Section 3, and refers back to it when necessary.

A ( $C^0$ ) *flow* is a continuous action  $f: \mathbb{R} \times X \rightarrow X$  of the additive reals on a certain topological space  $X$ . Each map  $f_t$  defined by  $f_t(x) = f(t, x)$  is a homeomorphism of  $X$ . A *fixed* or *stationary point* of a flow is one whose orbit,  $f(\mathbb{R} \times \{x\})$ , reduces to a point. Usually, flows without fixed point are referred to as *non-singular* or *non-stationary*. Yet it is convenient to compactify the jargon (we follow essentially Beck [6], modulo a slight compression):

**Definition 2.1** A flow with no stationary points is called a *brush*; and a space with a brush is a *brushing*.

Given a flow one may consider the partition into orbits, and expect—if both the space and the flow are sufficiently regular (say  $X$  a manifold and  $f$  a brush)—a sort of locally well-behaved geometric structure. This idea blossomed first to the notion of *Kurvenschar* or *regular family of curves* as defined by Kerékjártó [40], Kneser [42] and Whitney [71], and later to the concept of *foliation* of Ehresmann-Reeb (1944–1952). Thus, naively one would expect that the partition into orbits of a brush on a manifold produces a foliation. As we shall recall, this albeit correct in low-dimensions  $\leq 3$ , fails from dimension 4, upwards.

The other way around, given a one-dimensional orientable foliation one may ask for a compatible flow, whose orbits structure generates the given foliation. Though non-canonical this reverse procedure works in full generality (modulo the metrisability axiom).

Summing up, the situation is as follows:

(1) the canonical map from brushes to foliations is foiled in high-dimensions  $\geq 4$  (Bing-Chewning [19]), but well-defined in low-dimensions  $\leq 3$  (Whitney 1938 [72]), and this even in the non-metric case (chart orbit trick).

(2) Vice versa, the non-canonical map from (oriented) 1-foliations to brushes works unconditionally in the metric realm (Whitney 1933 [71]), but fails outside (Propositions 2.6 or 3.4).

### 2.1 The foliation induced by a brush: Whitney-Bebutov theory

Apart from detail of phraseology and a non-metric shift, the following is due to Whitney:

**Theorem 2.2** *Let  $f: \mathbb{R} \times M \rightarrow M$  be a non-singular  $C^0$ -flow on a (non-metric) manifold. Then the orbits of the flow induce a one-dimensional oriented foliation on  $M$ , provided (i) the flow is  $C^1$  or (ii) the manifold is of dimension  $n \leq 3$ .*

**Proof.** In the metric case the Whitney-Bebutov theory<sup>11</sup> ensures the existence, through any non-singular point of the flow, of a local cross-section and an associated flow-box. The metric proviso is in fact immaterial as choosing a Euclidean chart  $U$  around any point, the chart orbit  $f(\mathbb{R} \times U)$  is invariant and Lindelöf (hence metric). A foliated structure follows if one can establish the locally Euclidean character of the cross-section. Whitney 1933 [71] answers this question for  $n = 2$  by quoting a result of Hausdorff, while the case  $n = 3$  is treated in Whitney 1938 [72] via his 1932 characterisation of the 2-cell. ■

**Remark 2.3** Whitney [71, p. 259-260] asked in 1933: *given a regular family of curves filling a region in  $\mathbb{R}^n$  is there a cross-section through any point which is a closed  $(n-1)$ -cell?* In a related vein, O. Hájek asked (1968) at the end of his book [32, Problem 8, p. 225] (almost verbatim): *Decide whether or not every continuous dynamical system on a differential manifold is isomorphic to a differential system.* In 1974, Chewning [19] provided the negative answer by constructing a flow on  $\mathbb{R}^4$  induced from a *non-manifold factor* of  $\mathbb{R}^4$ , i.e. a non-locally Euclidean space  $X$  which crossed by  $\mathbb{R}$  becomes  $\mathbb{R}^4$ . (Such spaces, discovered about 1958 by Bing-Shapiro, arise by collapsing to a point a wild arc of  $\mathbb{R}^3$ .) Chewning's negative solution to Hájek's problem also answers the 1933 question of Whitney. Regarding *compact* transformation groups, non-smoothable actions were detected earlier in Bing 1952 [10], showing an exotic involution on  $\mathbb{S}^3$  by identifying the doubled Alexander solid horned sphere to  $\mathbb{S}^3$ . (In both cases the relative dimension of the action is 3.)

As in low dimensions  $\leq 3$  the relation from brushes to foliations is safe, we may deduce:

**Corollary 2.4** *The long plane  $\mathbb{L}^2$  lacks non-singular flows.*

**Proof.** Otherwise it would have a foliation by short leaves, violating the classification given in [4, Corollary 7.7]. An alternative (non-foliated) proof follows either from Theorem 4.5 (Poincaré-Bendixson approach) or from Corollary 3.19 (Brouwer's fixed-point theorem), which establishes the general case of  $\mathbb{L}^n$ . ■

## 2.2 Whitney's flows: creating motions compatible with a foliation

The shortest route from foliations back to dynamics is the following result of Whitney [71], whose 2-dimensional case goes back to Kerékjártó [41]. This is as follows, again apart from matters of phraseology (i.e., regular families of curves versus foliations):

**Theorem 2.5** *(Kerékjártó 1925, Whitney 1933) Given an orientable one-foliation on a metric  $C^0$ -manifold, there is a compatible flow whose orbits are the leaves of the foliation.*

**Proof.** For the case of the plane, see Kerékjártó [41, p. 111, §7]: looking at details it seems fair to say that he establishes the theorem in the 2-dimensional case (eventually with some assistance of Radó [63] to triangulate the surface). The general case is a 24 pages long ascension à la Whitney [71, Thm 27.A, p. 269]. (For more recent treatments compare eventually Mather [47] (surfaces) and Hector-Hirsch [33] (triangulations).) ■

In passing, we mention that, albeit limited to metric manifolds, this result of Whitney plays a crucial rôle in our previous classification of foliations on the long plane, for the asymptotic rigidity enables a reduction to certain compact subregions (see [4, Section 7] for the details).

## 2.3 Non-metric disruption of Whitney's flows (caused by Cantorian rigidity)

Of course Whitney's theorem (2.5) is trivially false in full generality, because as soon as there is a long leaf (e.g., the horizontal foliation of the long plane by long lines), the foliation cannot be the phase-portrait of a flow.

Thus, the more subtle question is whether Whitney's flows exist when all leaves are *short* (i.e., metric). This fails even when all leaves are compact (so circles) as shown by the following example. (Later we shall assist to another elementary disruption of Whitney's flows on the Moore surface, see Proposition 3.4.)

**Proposition 2.6** *The orientable foliation on  $\mathbb{L}^2 - \{0\}$  by concentric squares lacks a compatible flow.*

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<sup>11</sup>See Whitney [71, p. 260 and 270] and Bebutov 1940, Nemytskii-Stepanov [52, p. 333].



**Proof.** By contradiction, let  $f: \mathbb{R} \times M \rightarrow M$  be such a flow, where  $M = \mathbb{L}^2 - \{0\}$ . For any  $\alpha \in \mathbb{L}_+$ , let  $S_\alpha$  be the square of radius  $\alpha$  for the “long norm”, i.e.  $S_\alpha = \|\cdot\|^{-1}(\alpha)$ , where  $\|\cdot\|: M \rightarrow \mathbb{L}_+$  is defined by  $\|(x, y)\| = \max\{|x|, |y|\}$ . (The symbol  $|\cdot|: \mathbb{L} \rightarrow \mathbb{L}_{\geq 0}$  is the obvious “long absolute value”.) We define a map  $\tau: \mathbb{L}_+ \rightarrow \mathbb{R}$  taking each  $\alpha \in \mathbb{L}_+$  to the time  $\tau(\alpha)$  elapsed until the point  $(\alpha, 0) \in \mathbb{L}^2 - \{0\}$  returns to its initial position under the flow  $f$  (continuity is easy to check). Hence  $\tau$  must be *eventually constant*, say constant after some bound  $\beta \in \mathbb{L}_+$  (see e.g., [4, Lemma 4.3]). The ultimate stagnation of the period implies that the flow ultimately converts into an action of the circle  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  (assuming for simplicity  $\tau(\beta) = 1$ ). More precisely, if  $M_{\geq \alpha} = \|\cdot\|^{-1}([\alpha, \omega_1))$  denotes the part of  $M$  lying outside the square  $S_\alpha$ , then the restricted action of  $\mathbb{R}$  on  $M_{\geq \beta}$  descends to an action of  $\mathbb{S}^1$  on  $M_{\geq \beta}$ .

This action admits a “slice”,  $\Sigma = [\beta, \omega_1) \times \{0\}$ , i.e., the restricted action  $\psi: \mathbb{S}^1 \times \Sigma \rightarrow M_{\geq \beta}$  is a continuous bijection. For each  $\gamma \in \omega_1$ ,  $\psi$  restricts to a homeomorphism  $\psi_\gamma: \mathbb{S}^1 \times ([\beta, \gamma] \times \{0\}) \rightarrow \|\cdot\|^{-1}([\beta, \gamma])$  between each sublevel of the canonical  $\omega_1$ -exhaustion by compact annuli, hence the inverse map  $\psi^{-1}$  is continuous as well. (Alternatively one can deduce that  $\psi$  is a homeomorphism from Lemma 2.8.) This is impossible, for  $\psi$  relates two long pipes belonging to distinct topological types (cf. Lemma 2.7). ■

**Lemma 2.7** *The cylindric pipe  $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$  is not homeomorphic to the planar pipe  $\mathbb{L}^2 - (-1, 1)^2 =: \Pi$ .*

**Proof.** Specialists are certainly able to distinguish them by playing with embedded long rays (for a brief sketch see Nyikos [55, p. 670]). We find it however psychologically more relaxing, to argue in terms of foliated structures; by noticing that the planar pipe  $\Pi$  lacks a foliation by long rays transverse to the boundary. The proof (of this last statement) uses the methods in [4, Section 7]; we briefly recall the idea. By rigidity each “rhombic” quadrant (i.e., the results from cuts practiced along the four diagonals of  $\Pi$ ) is either asymptotically foliated by long straight rays or by short segments (cf. Figure 13 in [4]), giving a short list of 6 combinatorially distinct patterns. By suitable cuts, we may extract 6 different compact subregions (cf. Figure 14 in [4]). A plumbing argument (cf. again Figure 14) shows that all those 6 patterns (except the first) are actually impossible in view of the Euler obstruction). The first case left over is just a square whose boundary is a circle leaf, impeding the existence of a foliation of the specified type on  $\Pi$ . ■

**Lemma 2.8** *Let  $f: X \rightarrow Y$  be a continuous bijection. Assume that  $X$  is sequentially-compact and that  $Y$  is first countable and Hausdorff. Then  $f$  is a homeomorphism.*

**Proof.** We check the continuity of the inverse map  $f^{-1}$  by showing that  $f$  is closed. Let  $F$  be closed in  $X$ . By first countability of  $Y$  it is enough to check that  $f(F)$  is sequentially closed. Let  $y_n$  be a sequence in  $f(F)$  converging to  $y$ . Let  $x_n$  be the unique lift of  $y_n$  in  $F$ . By sequential-compactness there is a converging subsequence  $x_{n_k}$  converging to  $x \in F$ , say. By continuity, it follows that  $(y_n)$  converges to  $f(x)$ . By uniqueness of the limit in Hausdorff spaces, we have  $y = f(x)$ , hence  $y \in f(F)$ , as desired. ■

### 3 Flows via small propagation

By a *flow*, one understands a continuous group action  $f: \mathbb{R} \times X \rightarrow X$ , of the real line on a certain topological space. For the sake of geometric intuition, we shall primarily deal with the case where  $X$  is a (topological) manifold, *a priori* without imposing differentiability, nor metrisability. The driving idea in this section is to look how the simplest available point-sets in a manifold, namely *charts*, get sidetracked by the flow motion. Of special interest is the situation, where a chart (more generally a Lindelöf subset) has an orbit spreading all around the manifold (filling it completely) for in this case the *whole* manifold turns out to be Lindelöf, hence metric. This motivates the following jargon:

**Definition 3.1** A *propagator* for a flow  $f: \mathbb{R} \times X \rightarrow X$  is a subset  $\Sigma$  of the phase-space,  $X$ , such that the restricted map of the flow  $f: \mathbb{R} \times \Sigma \rightarrow X$  is surjective. The propagator is said to be *small* if it is Lindelöf. In that case the phase-space  $X$  is Lindelöf (for  $\mathbb{R} \times \Sigma$  is Lindelöf, as  $\mathbb{R}$  is  $\sigma$ -compact).

#### 3.1 Extinction of minimal flows on non-metric manifolds

The following is a very baby non-metric version of the Gottschalk conjecture [27], inquiring which spaces, especially manifolds, carries a minimal flow:

**Proposition 3.2** *A manifold carrying a minimal flow is metric.*

**Proof.** In a minimal flow, each non-empty open set is a propagator. ■

**Remark 3.3** (*Minimal foliations via Nyikos’ cytoplasmic expansions*) In contrast, it is possible to construct minimal foliations on non-metric surfaces, via a Kronecker (irrational slope) foliation on the 2-torus. Recall a remarkable construction of Nyikos [56], attaching a long ray to certain surfaces. The plane  $\mathbb{R}^2$ , for instance, can be stretched in one or more directions to give “amoebas” with long *cytoplasmic expansion(s)* of the 2-cell (by so-called *pseudopodia*). When applied to a punctured Kronecker torus, such a long cytoplasmic expansion—effected near the puncture in a way parallel to the foliation—produces a minimally foliated non-metric surface with one long-ray leaf. Besides, by a clever Prüferisation along a suitable Cantor set in the torus, M. Baillif also proposes a minimal foliation having only metric leaves (details may appear in a subsequent paper Baillif *et al.* [5]).

### 3.2 Disruption of Whitney’s flows detected by small propagation

The idea of propagation is now used to identify more examples where Whitney’s flows run into troubles, e.g. on the *Moore surface*. Recall, the latter to be deduced from the bordered Prüfer surface,  $P$ , by self-identifying the boundary components via the folding  $x \sim -x$ . (For the original description cf. [49] and the 1962 edition of the *Foundations* [50, p.376–377].) The horizontal foliation on  $P$  when pushed down to  $M$  develops many *thorns* singularities (cf. Figure 1). Thus, we rather consider the vertical “foliation” on  $P$  ( $x = \text{const}$ ), which in fact is *not* a genuine foliation, for it has *saddle* singularities (locally equivalent to the level curves of the function  $xy$ , restricted to the upper-half plane). Yet, the quotient mapping  $P \rightarrow M$  resolves these singularities to induce a (genuine) foliation on the Moore surface  $M$ , which we refer to as the *vertical foliation* of  $M$ .

**Proposition 3.4** *The vertical foliation on the Moore surface,  $M$ , lacks a compatible Whitney flow.*

**Proof.** Notice that the horizontal line  $y = 1$  is a small propagator for any compatible flow, violating the non-metrisability of the Moore surface. ■

This exemplifies probably one the simplest manifestation of the radical divorce affecting (outside the metric sphere) the fusional cohesion of “flows” with “foliations”, under Whitney’s marriages.

### 3.3 Prüfer’s flows: windscreen wiper motions

Small propagators are not always available, e.g., in the case of the horizontal foliations on the Prüfer surfaces (either  $2P$  or  $P_{\text{collar}}$ ). The natural candidates would be countable unions of vertical lines, but then some of the uncountably many *bridges* (cf. Figure 1, for an intuitive meaning) remain unexplored. Thus, the “flowability” of these foliations is not obstructed by small propagation, and it turns out that those horizontal foliations on the varied Prüfer surfaces admit indeed a Whitney flow. Heuristically, such a flow (on  $P$ ) can be visualised as the motion of a “windscreen wiper”, where the different rays (involved in Prüfer) are undergoing a collective sweeping motion (providing thereby a first interesting motion (brush) on a non-metric manifold, not merely induced by a metric factor). Here is a formal treatment:

**Proposition 3.5** *The horizontal foliation on the bordered Prüfer surface  $P$  admits a compatible Whitney flow.*

**Proof.** Such a flow  $f: \mathbb{R} \times P \rightarrow P$  is obtained as follows: given  $p \in P$  and  $t \in \mathbb{R}$  there are two cases:

(i) If  $p$  is a “genuine” point (i.e., not a “ray”) then  $p = (x, y)$ , and define  $f(t, (x, y)) = (x + yt, y)$ . [Geometrically, given  $p$ , draw the vertical line through  $p$  and take its intersection with the  $(y = 1)$ -line, to which a horizontal translation of amplitude  $t$  is operated (to get the point  $q = (x + t, 1)$ ), and  $f(t, p)$  is the intersection point of the line through  $(x, 0)$  and  $q$  with the horizontal line at height  $y$ .]

(ii) If  $p$  is a ray emanating from  $(x, 0)$ , then  $p$  has some “slope”  $s$  defined as  $\frac{\Delta x}{\Delta y}$ , and  $f(t, p)$  is defined as the ray through  $(x, 0)$  of slope  $s + t$ .

Both prescriptions are consistent: consider a sequence  $p_i$  of points converging to a ray through  $(x_0, 0)$  of slope  $s$ . The ray has an equation  $x = sy + x_0$ , and we may assume that the  $p_i = (x_i, y_i)$  lie (eventually) on this ray while converging to it. Then  $f(t, p_i) = (x_i + y_i t, y_i) = (sy_i + x_0 + y_i t, y_i) = ((s + t)y_i + x_0, y_i)$ , showing that the  $f(t, p_i)$ ’s converge to the ray of slope  $s + t$  through  $(x_0, 0)$ .

Finally the group property holds:  $f(t', f(t, (x, y))) = f(t', (x + yt, y)) = (x + yt + y t', y) = (x + y(t + t'), y) = f(t + t', (x, y))$ , while for rays the property is obvious. ■

**Example 3.6** Such Prüfer flows also exist on the 2-torus irrationally foliated and Prüferised along a closed interval in a leaf. (This is made more explicit in case (16) of Section 5.3.) If the Prüferisation is effected so that it “locally” resembles the doubled Prüfer  $2P$  (leading to a separable surface), then the Prüfer flow is *transitive* (i.e., exhibit at least one dense orbit). In fact, only the points lying on the “bridges” lack a dense orbit.

What is the minimal cardinality of non-dense orbits for a flow on a non-metric manifold? Small propagation again gives a quick (Cantor relativistic) answer:

**Proposition 3.7** *A flow on a non-metric manifold has uncountably many non-dense orbits.*

**Proof.** Otherwise a small propagator is designed by aggregating to a chart countably many points picked in the non-dense orbits. ■

Thus, the minimal cardinality in question is at least  $\omega_1$  (and at most  $\mathfrak{c} = \text{card}\mathbb{R}$ , for the cardinality of a connected Hausdorff  $n$ -manifold is  $\mathfrak{c}$ , provided  $n \geq 1$  [67], [55, Thm 2.9]). The Prüfer flow of Example 3.6 has exactly  $\mathfrak{c}$  many non-dense orbits, hence realises the lower-bound under the continuum hypothesis (CH). If the negation of (CH) holds (i.e.,  $\omega_1 < \mathfrak{c}$ ), we may throw away non-dense orbits until precisely  $\omega_1$  are left. In conclusion the answer is  $\omega_1$ , yet the “exact” size of  $\omega_1$  depends on (CH).

### 3.4 Cytoplasmic obstruction to non-singular flows

Since non-metric manifolds lack the best possible *minimal* dynamics, we switch to the weaker paradigm of non-singular flows (alias *brushes*), by wondering which manifolds can support them?

From (3.5), the Prüfer surfaces (in all its three incarnations:  $P$ ,  $2P$  and  $P_{\text{collar}}$ ) all admit a brush. By contrast, we know already one non-metric surface lacking a brush, namely the long plane  $\mathbb{L}^2$  (2.4). Regarding the Moore surface, we noticed in (3.4) that the natural vertical foliation lacks a compatible brush (a priori not excluding the existence of a brush inducing a more exotic foliation). Yet, no such exotica with Moore:

**Proposition 3.8** *The Moore surface lacks any brush.*

**Proof.** The Moore surface consists of a “core” corresponding to the interior of  $P$ , plus a continuous collection of thin “thorns” hanging on (picturesquely like thin stalactites), cf. Figure 1. This very specific morphology implies the core to be a propagator under any brush, for any point on a thorn (homeomorphic to a semi-line  $\mathbb{R}_{\geq 0}$ ) will eventually flow into the core. ■

This argument readily generalises to the following class of surfaces (indeed manifolds of arbitrary dimensionality) abstracting the morphology of the Moore surface. (This structure—which we shall refer to as *cytoplasmic*—may perhaps be regarded as a “separable” counterpart to the *bagpipe* structure of Nyikos [55] concretising the point-set paradigm of  $\omega$ -boundedness):

**Definition 3.9** A manifold,  $M$ , is said to admit a *cytoplasmic* (or *core-thorn*) *decomposition* (both abridged CTD) if it contains an open set  $U$  which is Lindelöf (the *core*) such that the residual set  $M - U$  decomposes into connected components as  $\bigsqcup_{x \in X} T_x$ , where each *thorn*  $T_x$  is a bordered one-manifold with a single boundary-point, so homeomorphic either to  $\mathbb{R}_{\geq 0}$  or to  $\mathbb{L}_{\geq 0}$ . (The unique boundary point of each thorn is used as indexing parameter.) Further we assume the following axiom:

(CT) Each point  $z \in T_x$  on a thorn admits a fundamental system of open neighbourhoods  $U_z$  consisting of points on the thorn plus some non-empty set of point in the core.

The following properties are easily verified:

(CT1) For each thorn  $T_x$ , the set  $U \cup T_x$  is open.

(CT2) The core  $U$  is dense in  $M$ , and being Lindelöf (hence separable, since manifolds are locally second countable) it follows that  $M$  is separable.

Surfaces tolerating a cytoplasmic decomposition include the Moore and the *Maungakiekie surface*<sup>12</sup>. The latter refers to the result of a unique cytoplasmic expansion of an open 2-cell by a long ray, as discussed in (3.3). To mention an example in dimension 3, one may consider the 3-dimensional avatar of the bordered Prüfer surface, say  $P^3$  (likewise constructed from the half-3-space by adjunction of an ideal boundary of “rays”) and fold the boundary-components (homeomorphic to  $\mathbb{R}^2$ ) by collapsing all points on concentric circles.

**Example 3.10** (*Prüferisation and Moorisation*) For more (including non simply-connected) examples, one needs only to plagiarise the Prüfer construction over any metric bordered surface,  $W$ , (e.g., an annulus  $\mathbb{S}^1 \times [0, 1]$ ), producing a non-metric bordered surface,  $P(W)$ , called the *Prüferisation* of  $W$ . Then, folding the boundaries gives a *Moorisation*,  $M(W)$ . The latter is separable and comes with a (canonic) cytoplasmic decomposition. (For more details, compare Definition 5.8.)

**Theorem 3.11** *A non-metric manifold with a cytoplasmic decomposition  $M = U \sqcup \bigsqcup_x T_x$  has no brush.*

<sup>12</sup>So called after a certain hill in New Zealand surmounted by a thin tower.

**Proof.** It is again the matter of observing that the core is a small propagator for any brush on  $M$ . ■

Corollaries of the theorem includes:

**Corollary 3.12** (i) *The Moore surface (and all its avatar  $M(W)$  in (3.10)), as well as the Maungakiekie (and a myriad of other “protozoans” deduced via cytoplasmic expansions) are not brushings.*  
(ii) *The doubled Prüfer surface,  $2P$ , as it is a brushing (3.5), does not tolerate a cytoplasmic decomposition.*

The failure of the Moore surface to brush suggests the following question (which will be answered as Theorem 4.18):

**Question 3.13** (*Pseudo-Moore problem*) *A pseudo-Moore surface—defined as one satisfying the following four axioms (verified by the classic Moore surface): (i) simply-connected, (ii) separable, (iii) non-metric, (iv) without boundary—lacks a brush.*

Relaxing any of the four assumptions foils the conclusion: If (i) is relaxed take  $2P$ , if (ii) is omitted  $\mathbb{R} \times \mathbb{L}$  works, if (iii) is suppressed  $\mathbb{R}^2$  is fine, and if (iv) is left  $P$  is suitable. The two-dimensionality is also crucial, since the 3-manifold  $\mathbb{R} \times (\text{Moore})$  has a brush.

Our initial motivation for cytoplasmic decompositions was an attempt to solve the above Pseudo-Moore problem. (Speculating that any pseudo-Moore surface permits a CTD, the problem reduces to Theorem 3.11.) Our later solution follows an entirely different route—at least in appearances—using more geometric methods (non-metric Schoenflies and phagocytosis). Yet, by analogy with the ubiquity of Nyikos’ bagpipe structure (in the 2-dimensional realm at least), it sounds natural to conjecture:

**Conjecture 3.14** *Any separable 2-manifold with countable fundamental group has a cytoplasmic decomposition.*

The proviso on the fundamental group is of course essential (else  $2P$  is a counterexample). The more intrinsic reason is of course that the condition looks necessary: for, given a cytoplasmic structure on a surface  $M$ , it is likely that the natural morphism  $\pi_1(U) \rightarrow \pi_1(M)$  induced by the “core” inclusion is isomorphic (we leave this in *une ombre propice*<sup>13</sup>, since we shall not use it). At any rate, a dynamical consequence of (3.14) is that any such surface, if non-metric, lacks a brush in view of (3.11). This assertion will be proved later (Corollary 4.19), via a trivial reduction to the simply-connected case (passage to the universal covering). In case Conjecture 3.14 should be true (and of real independent interest), then it is quite likely that the proof will depend on techniques à la Morton Brown (cf. especially what we call the *inflation conjecture* discussed later as item (5.15): this is susceptible to produce the “core” via inflation of an open set covering faithfully a basis of the  $\pi_1$ , after using Schoenflies to fill inessential circles).

### 3.5 Dynamical Euler characteristic: Cartesian multiplicativity?

Now, a little curiosity: define the (*dynamical*) *characteristic*  $\delta(M) \in \mathbb{F}_2$  (the field with two elements) of a manifold  $M$  as equal to 0 if  $M$  has a brush, and 1 otherwise.

**Question 3.15** *Is  $\delta$  multiplicative under Cartesian products, i.e.  $\delta(M \times N) = \delta(M) \cdot \delta(N)$ ?*

The formula is obvious if one of the two factors has vanishing  $\delta$ -characteristic (one can brush the product from one of its factor). If “multiplicativity” holds it would imply  $\delta(\mathbb{L}^n) = 1$  (and  $\delta(\mathbb{L}_+^n) = 1$ ). This reminds the question of Kuratowski as to whether the product of two spaces having the fpp (fixed point property) still has the fpp: the failure is well-known (cf. R.F. Brown’s survey [14]). The case  $n = 2$  of  $\delta(\mathbb{L}^n) = 1$  follows from the classification of foliations on  $\mathbb{L}^2$  (2.4). Maybe, the square of the Moore surface,  $M \times M$ , provides a negative answer to (3.15): for  $M \times M$  may have a brush, since the product of thorns  $T_x$  are quadrants  $\mathbb{R}_{\geq 0}^2$  which tolerate a brush (not constrained to move into the core).

### 3.6 Fixed point property for flows via sequential-compactness: the case of $\mathbb{L}^n$

The general case of  $\delta(\mathbb{L}^n) = 1$  can be proven independently of (3.15) via the:

**Proposition 3.16** *Let  $M$  be a sequentially-compact manifold with the fpp (fixed point property). Then  $\delta(M) = 1$ , i.e.  $M$  has the fpp for flows. (In fact “manifold” can be replaced by “Hausdorff space”, and it is enough to assume fpp for homeomorphisms isotopic to the identity.)*

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<sup>13</sup>Well-known phraseology of René Thom.

**Proof.** This is the standard<sup>14</sup> “dyadic cascading” argument. Let  $f: \mathbb{R} \times M \rightarrow M$  be any flow. Write  $f_t$  for the time- $t$  map, i.e.  $f_t(x) = f(t, x)$ . Let  $t_n = 1/2^n$  for  $n$  an integer  $\geq 0$  and let  $K_n$  be the fixed-point set of  $f_{t_n}$ . The sets  $K_n$  are closed ( $M$  is Hausdorff), non-empty ( $M$  has the fpp) and nested  $K_n \supset K_{n+1}$  (as follows from the group property). Next, observe the:

**Lemma 3.17** *If  $(K_n)_{n \in \omega}$  is a nested sequence of non-empty closed sets in a sequentially-compact space  $X$ , then the infinite intersection  $\bigcap_{n \in \omega} K_n$  is non-empty.*

**Proof.** Choose, for each  $n \geq 0$ , a point  $x_n \in K_n$ . By sequential-compactness the sequence  $(x_n)$  has a converging subsequence, whose limit  $x$  belongs to the intersection of all  $K_n$ ’s. ■

By (3.17) there is a point in  $\bigcap_{n \geq 0} K_n$ , thus fixed under all dyadic times, hence under all times. ■

It remains to notice a transfinite avatar of the Brouwer fixed point theorem:

**Lemma 3.18** *Let  $\mathbb{L}$  be the long line, then  $\mathbb{L}^n$  has the fixed point property for all  $n$ .*

**Proof.** (Communicated by Mathieu Baillif.) Let  $f: \mathbb{L}^n \rightarrow \mathbb{L}^n$  be continuous, and let  $U_\alpha$  be  $(-\alpha, \alpha)^n \subset \mathbb{L}^n$  for  $\alpha \in \omega_1$ . Since  $U_\alpha$  is Lindelöf,  $f(U_\alpha)$  is contained in  $U_{\beta(\alpha)}$  for some  $\beta(\alpha) \geq \alpha$ . Set  $\alpha_0 = 1$  and  $\alpha_n = \beta(\alpha_{n-1})$  for  $n \in \omega$ . Let  $\alpha$  be the supremum of the  $\alpha_n$ ’s, then  $f(U_\alpha) \subset U_\alpha$ . By continuity,  $f(\overline{U_\alpha}) \subset \overline{f(U_\alpha)} \subset \overline{U_\alpha}$ , where  $\overline{U_\alpha} = [-\alpha, \alpha]^n$  is a topological ball, and we conclude via the Brouwer fixed point theorem. ■

**Corollary 3.19** *The long hyperspace  $\mathbb{L}^n$  has the fixed-point property for flows for all integer  $n \geq 0$ .*

This method has a wider flexibility, for it applies to other (balls-exhaustible) manifolds and one may also harpoon the fixed-point via Lefschetz, instead of Brouwer (these ideas are discussed in Section 4.3).

## 4 Dynamical consequences of Schoenflies

We now address certain dynamical repercussions of the “Schoenflies axiom” (any embedded circle bounds a disc). In view of [23], this amounts to simple-connectivity. This section certainly contains the most satisfactory results of the paper, inherited from the “tameness” of 2-dimensional—and the allied plane—topology. This explains why the classical paradigms of 2-dimensional dynamics transpose so easily to the non-metric realm (without having to control the growing mode of the manifold). As we shall oft deplore later, nothing similar seems to occur in higher-dimensions (and to be perfectly honest, not even in 2-dimensions when the fundamental group is allowed to be non-trivial—recall the plague of wild pipes).

### 4.1 Poincaré-Bendixson theory (dynamical consequence of Jordan)

Our first objective is Theorem 4.5 below, whose formal proof requires some preparatory lemmata, all very standard. For the sake of short-circuiting some logical deductions by geometric intuition here is a quick justification of (4.5): by  $\omega$ -boundedness the surface is “long” impeding a “short” semi-orbit to escape at infinity. Thus, any trajectory begins a spiraling motion, creating asymptotically either a stationary point or a periodic orbit. In the latter case, Schoenflies gives an invariant bounding disc, where a fixed point is created by Brouwer. *q.e.d.*

The rest of this subsection details the more pedestrian route (thus skip it, if you like):

**Lemma 4.1** *A flow on a Schoenflies surface with a periodic orbit has a fixed-point. (Same conclusion if the periodic orbit is null-homotopic.)*

**Proof.** The periodic orbit is a topological circle, so bounds a 2-disc, which is invariant under the flow. Applying the Brouwer fixed point theorem to the dyadic times maps  $f_{t_n}$  of the flow ( $t_n = 1/2^n$ ), we get a nested sequence of non-empty fixed-point sets  $K_n = \text{Fix}(f_{t_n})$  whose common intersection  $\bigcap_{n=1}^{\infty} K_n$  is non-empty (compactness of the disc). By continuity, a point in this set is fixed under the flow. ■

Next, we seek for an intrinsic topological property forcing the formation of periodic orbits. A good condition is  $\omega$ -boundedness<sup>15</sup>: for given a flow  $f$ , the set  $\mathbb{Q}x = f(\mathbb{Q} \times \{x\})$  is countable, hence its closure  $\overline{\mathbb{Q}x}$  is compact. Note that  $\overline{\mathbb{Q}x} = \overline{\mathbb{R}x}$  (as follows from the general formula  $f(\overline{S}) \subset \overline{f(S)}$  for a continuous map  $f$ ). A periodic motion is produced by the following standard mechanism (of G. D. Birkhoff):

<sup>14</sup>Compare, e.g., Lima 1964 [44, Lemma 4, p. 101], Hector-Hirsch [33], Vick [69, Thm 7.28, p. 208].

<sup>15</sup>Recall that a space is  $\omega$ -bounded if each countable subset has a compact closure.

**Lemma 4.2** *Let  $f$  be a flow on a (Hausdorff) space  $X$ . Assume that the orbit closure  $\overline{\mathbb{R}x}$  is compact and the flow proper (i.e. each “inherent” open set of the form  $f([s, t] \times \{z\})$  is also open for the relative topology on  $\mathbb{R}z$ ). Then there is a compact orbit, which by properness is either a fixed point or a periodic orbit.*

**Proof.** Consider the set  $\Sigma$  of all non-empty closed  $f$ -invariant subsets of the compactum  $\overline{\mathbb{R}x}$ . Order it by inclusion. Inductiveness follows from the non-emptiness of a nested intersection of closed sets in a compactum. By Zorn’s lemma, there is  $K$  a minimal set in  $\Sigma$ .

We show that  $K$  reduces to a single orbit. Indeed let  $y \in K$ , then  $\mathbb{R}y \subset \overline{\mathbb{R}y} \subset K$ , and the last inclusion is an equality by minimality of  $K$ . So it is enough to check that  $\mathbb{R}y$  is closed. If not then  $\mathbb{R}y - \mathbb{R}y$  is non-empty, invariant and closed (by Lemma 4.3 below), contradicting minimality. It remains to check:

**Lemma 4.3** *Let  $f$  be a flow on a (first countable and Hausdorff) space  $X$ . If the orbit  $\mathbb{R}x$  is proper, then the set  $\overline{\mathbb{R}x} - \mathbb{R}x$  is closed (in  $X$ ).*

**Proof.** It is enough to show that  $\mathbb{R}x$  is open in  $\overline{\mathbb{R}x}$ . Let  $z \in \mathbb{R}x$ . Choose any  $\varepsilon > 0$ , then  $f([- \varepsilon, \varepsilon] \times \{z\})$  is open for the inherent topology on  $\mathbb{R}x$ , so by properness there is  $U$  open in  $X$  such that  $U \cap \mathbb{R}x = f([- \varepsilon, \varepsilon] \times \{z\})$ . Then  $U \cap \overline{\mathbb{R}x}$  is open in  $\overline{\mathbb{R}x}$ , contains  $z$  and satisfies  $U \cap \overline{\mathbb{R}x} \subset \mathbb{R}x$ .

Indeed let  $y \in U \cap \overline{\mathbb{R}x}$ . Since  $X$  is first countable, we choose an approximating sequence  $y_n \in \mathbb{R}x$  converging to  $y$ . Since  $U$  is open, there is an integer  $N$  such that  $y_n \in U$  for all  $n \geq N$ . So  $y_n \in U \cap \mathbb{R}x$ , hence  $y_n = f(t_n, z)$  with  $|t_n| < \varepsilon$ . But since  $f([- \varepsilon, \varepsilon] \times \{z\}) =: F$  is compact (hence closed in  $X$  Hausdorff), we have that  $y \in F$ . Since  $F \subset \mathbb{R}z = \mathbb{R}x$ , this completes the proof. ■

Summing up,  $K$  is a compact orbit, and properness implies that orbits are connected Lindelöf  $n$ -manifolds with  $n \leq 1$ , so that  $K$  is either a point or a circle. ■

For surfaces, properness is ensured by the *dichotomy* of the underlying surface (i.e., Jordan separation by circles holds true):

**Lemma 4.4** *Every flow on a dichotomic surface is proper.*

**Proof.** (The classical Bendixson sack argument.) Given an inherent open set  $I_\varepsilon := f([- \varepsilon, \varepsilon] \times \{x\})$  we seek an open set  $U$  such that  $U \cap \mathbb{R}x = I_\varepsilon$ . If the point  $x$  is stationary,  $U$  is easy-to-find. If non-stationary the theory of flow-boxes is available. In this theory, metrisability plays a crucial rôle, which turns out however to be subsidiary; for letting flow any chart domain  $V$  about  $x$  yields the set  $f(\mathbb{R} \times V)$  which is Lindelöf, hence metric. According to Bebutov (cf. Nemytskii-Stepanov [52, p.333–335]) a flow-box can be found for any preassigned time length, shrinking eventually the cross-section  $\Sigma_x \ni x$ . Inside the flow-box  $B := f([- \varepsilon, \varepsilon] \times \Sigma_x)$  a certain sub-rectangle  $R$  is singled out by the first returns of  $x$  to the section  $\Sigma_x$  (both forwardly and backwardly in time). Dichotomy ensures via a Bendixson sack argument the absence of further “recurrences”, i.e. subsequent returns intercepting the section  $\Sigma_x$  closer to  $x$ . The existence of the desired  $U$  is guaranteed (the interior of  $R$  is appropriate). ■

## 4.2 $\omega$ -bounded hairy ball theorem

Assembling the previous facts, we obtain:

**Theorem 4.5** *Any flow on a  $\omega$ -bounded Schoenflies surface (equivalently simply-connected) has a fixed point.*

**Proof.** Since simply-connected implies dichotomic [23], the flow is proper by (4.4). Take any point  $x \in M$ , its orbit closure  $\overline{\mathbb{Q}x} = \overline{\mathbb{R}x}$  is compact, by  $\omega$ -boundedness. By (4.2), the flow has either a fixed point or a periodic orbit. In the latter case one applies (4.1). ■

This may be regarded as a non-metric avatar of the “hairy ball theorem” (Poincaré, Dyck, Brouwer): the 2-sphere cannot be foliated nor brushed. The theorem applies for instance to the long plane  $\mathbb{L}^2$  (which case also follows from the classification of foliations on  $\mathbb{L}^2$  given in [4]). It also applies to any space obtained from a Nyikos long pipe, [55], by capping off the short end by a 2-disc, for example the *long glass*, i.e. the semi-long cylinder  $\mathbb{S}^1 \times (\text{closed long ray})$  capped off by a 2-disc. The proof just given also shows:

**Corollary 4.6** *Any flow on a  $\omega$ -bounded dichotomic surface has either a fixed point or a periodic orbit.*

This applies to any surface deduced from the 2-sphere by insertion of a finite number of long pipes. (For instance any brush on the long cylinder  $\mathbb{S}^1 \times \mathbb{L}$  has a periodic motion.)

### 4.3 Fragmentary high-dimensional hairy ball theorems via Lefschetz

It seems natural to wonder if Theorem 4.5 generalises to dimension 3 (and higher). A basic corruption is the (compact) 3-sphere  $\mathbb{S}^3$  brushed via a (Clifford-)Hopf fibering ( $\mathbb{S}^1$ -action arising from the ambient complex coordinates). By the Poincaré conjecture this is the only compact counterexample. The following speculates this to be the only failure in general:

**Conjecture 4.7** *Any simply-connected  $\omega$ -bounded non-metric 3-manifold has the fpp for flows.*

The same in dimension  $n \geq 4$  is foiled: consider  $\mathbb{S}^3 \times \mathbb{L}^{n-3}$  brushed along the first factor by the Hopf fibering. Let us try to collect some experimental evidence towards (4.7). First, the assertion holds for  $\mathbb{L}^3$  (Corollary 3.19). By the proof of Lemma 3.18, the conjecture holds more generally if the 3-manifold admits an  $\omega_1$ -exhaustion by (compact) 3-balls (at least if the interior exhaustion is *canonical*—in the sense of Nyikos discussed below). However, this fails to cover the general case, as  $\mathbb{S}^2 \times \mathbb{L}$  lacks a ball-exhaustion having a non-trivial second homotopy group  $\pi_2$ . To handle this situation we use Lefschetz in place of Brouwer’s fixed point theorem, to obtain:

**Proposition 4.8** *Let  $M$  be an  $\omega$ -bounded  $n$ -manifold. Assume the existence of a canonical exhaustion  $M = \bigcup_{\alpha < \omega_1} W_\alpha$  by compact bordered<sup>16</sup>  $n$ -submanifolds  $W_\alpha$  with non-vanishing Euler characteristic. Then any flow on  $M$  has a fixed point.*

**Proof.** By the cascading argument in the proof of (3.16), it is enough to show that  $M$  has the fpp for a map  $f$  homotopic to the identity (in fact, sequential compactness follows from the exhaustion assumption, so that we could relax “ $\omega$ -boundedness”). Lemma 4.9 below produces some big  $\alpha \in \omega_1$  so that  $f(W_\alpha) \subset W_\alpha$ , and with corresponding restriction  $f_\alpha = f|_{W_\alpha}: W_\alpha \rightarrow W_\alpha$  homotopic to the identity. The hypothesis  $\chi(W_\alpha) \neq 0$  implies via the Lefschetz fixed point theorem (for compact metric ANR’s [43]) that  $f_\alpha$  has a fixed point. ■

**Lemma 4.9** *Let  $M = \bigcup_{\alpha < \omega_1} M_\alpha$  be a space with an  $\omega_1$ -exhaustion by Lindelöf open subsets  $M_\alpha$  verifying the continuity axiom  $M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$  whenever  $\lambda$  is a limit ordinal.*

- (i) *Given a continuous map  $f: M \rightarrow M$  there is a club<sup>17</sup>  $C$  of indices  $\alpha$  such that  $f(M_\alpha) \subset M_\alpha$ .*
- (ii) *Moreover if  $f$  is homotopic to the identity of  $M$ , then there is some  $\alpha \in \omega_1$  so that the restriction  $f: \overline{M_\alpha} \rightarrow \overline{M_\alpha}$  is homotopic to the identity.*

**Proof.** (i) The image  $f(M_\alpha)$  is Lindelöf, hence there is a  $\beta(\alpha) \geq \alpha$  so that  $f(M_\alpha) \subset M_{\beta(\alpha)}$ . Define inductively  $\alpha_1 = 1$ ,  $\alpha_n = \beta(\alpha_{n-1})$  and let  $\alpha = \sup_n \alpha_n$ . By the continuity axiom  $M_\alpha = \bigcup_{n=1}^\infty M_{\alpha_n}$ . Consequently,  $f(M_\alpha) = f(\bigcup_{n=1}^\infty M_{\alpha_n}) = \bigcup_{n=1}^\infty f(M_{\alpha_n}) \subset \bigcup_{n=1}^\infty M_{\alpha_{n+1}} = M_\alpha$ . This implies the first clause.

(ii) Let  $(f_t)$ ,  $t \in [0, 1]$  be a homotopy relating  $\text{id}_M = f_0$  to  $f = f_1$ . Let  $D$  be the set of dyadic numbers in  $[0, 1]$ . For each  $t \in D$ , the map  $f_t$  preserves a club  $C_t$  of stages  $M_\alpha$  by (i). As any countable intersection of clubs is again a club<sup>18</sup>,  $C = \bigcap_{t \in D} C_t$  is a club indexing stages preserved by all dyadic-times of the homotopy  $(f_t)$ , i.e.  $f_t(M_\alpha) \subset M_\alpha$  for all  $t \in D$  and all  $\alpha \in C$ . By continuity  $f_t(\overline{M_\alpha}) \subset \overline{f_t(M_\alpha)} \subset \overline{M_\alpha}$ . By joint-continuity, the inclusions  $f_t(\overline{M_\alpha}) \subset \overline{M_\alpha}$  (for  $\alpha \in C$ ) hold indeed for all  $t \in [0, 1]$ . This provides the required homotopy. ■

Typical illustrations of Proposition 4.8 arise from the 3-sphere by excising a finite number  $n \geq 1$  of (tame) 3-balls, and then capping off by long 3D-pipes, e.g. of the form  $\mathbb{S}^2 \times \mathbb{L}_{\geq 0}$ . (Other pipes can be used, provided they are sufficiently “civilised” to allow an exhaustion of the bordered canonical type.) Thus, all such 3-manifolds have the fpp for flows, provided  $n \geq 1$ .

**Remark 4.10** By the Poincaré conjecture, the above family of examples is essentially exhaustive. Indeed, a simply-connected bordered compact 3-manifold  $W$  is a holed 3-sphere, i.e.,  $\mathbb{S}^3$  excised by the interior of a disjoint finite family of tame 3-balls. [Proof: by duality the boundary components of  $W$  are 2-spheres, cap them off by 3-balls to apply the Poincaré conjecture. Decapsulating back the added 3-balls leads to the conclusion.] Therefore, the 3-ball and its holed avatars (*Swiss cheeses* or *Grundformen* in the jargon of Möbius [48]) are the only possible bags candidates, for the construction of simply-connected  $\omega$ -bounded 3-manifolds.

<sup>16</sup>Recall that, *bordered manifold* is understood as a synonym of manifold-with-boundary. Moreover “canonical” refers to the hypothesis that the “interiorised” exhaustion  $M = \bigcup_{\alpha < \omega_1} \text{int}(W_\alpha)$  by the interiors of the  $W_\alpha$ ’s satisfies the continuity axiom of Lemma 4.9 right below.

<sup>17</sup>Closed unbounded subset of  $\omega_1$ .

<sup>18</sup>This is an exercise related to the leapfrog argument. Hint: the leapfrog argument shows the case of a two-fold intersection. Induction the case of a finite intersection. The countable case intersection easily follows: if  $C_n$ ,  $n \in \omega$  is a countable collection of clubs, choose  $x_1 \in C_1$ , find a greater  $x_2 \in C_1 \cap C_2$ , and so on  $x_n \in C_1 \cap \dots \cap C_n$  greater than  $x_{n-1}$ , then  $\sup_{n < \omega} x_n$  belongs to all  $C_n$ ’s. *q.e.d.*

Albeit far from proving it, Proposition 4.8 infers a certain evidence to Conjecture 4.7, in the sense that any corruption—if it exists—is instigated by a pipe lacking a nice canonical exhaustion—a *wild* pipe, so to speak.

Even though Conjecture 4.7 fails in dimension  $\geq 4$ , Proposition 4.8 gives varied special cases, e.g.,  $\mathbb{S}^2 \times \mathbb{L}^2$  (or more generally  $M \times \mathbb{L}^n$ , where  $M$  is a closed manifold with  $\chi(M) \neq 0$  and  $n \geq 0$ ), for those manifolds have a regulated growing mode (canonical exhaustion) around a bag with non-zero Euler characteristic.

**Remark 4.11** Of course Lefschetz leads to somewhat stronger results than those obtained via Brouwer, even when each  $W_\alpha$ 's are contractible. For instance, Proposition 4.8 applies to the contractible manifold-patch  $W^4$  (of Freedman 1982 [22]) bounding the Poincaré homology 3-sphere  $\Sigma^3$ , considered as the bag of the manifold obtained by capping off the boundary by  $\Sigma^3 \times \mathbb{L}_{\geq 0}$ .

**Remark 4.12** ( *$\omega$ -bounded disruption of Heinz Hopf*) One should not expect too much from a reverse engineering to Proposition 4.8, where assuming each  $W_\alpha$  brushed one hopes by a transfinite assembly to gain a brush on  $M$ . For instance, on the connected-sum surface  $M = \mathbb{L}^2 \# \mathbb{L}^2$  (two gemelar long planes  $\mathbb{L}^2$  linked by a worm hole), one cannot glue transfinitely the natural flows on the annuli of the canonical  $\omega_1$ -exhaustion. This would corrupt the global topology of the pipe (compare Proposition 2.6). With some extra work, one can show that  $M$  lacks any brush. This involves a special argument similar to the one exposed in Proposition 4.16 below.

**Remark 4.13** Proposition 4.8 applies as well in dimension 2, offering an alternate road to Theorem 4.5, at first glance at least. However this approach is again plagued by the existence of “wild” pipes (cf. Nyikos [55, §6, p.669–670]). Hence the Poincaré-Bendixson argument (Theorem 4.5) still incarnates a wider range of applicability, for it applies unconditionally (yet, just in the simply connected case!).

The following two hazardous conjectures (the first would follow from the second) are merely an avowal of our ignorance:

**Conjecture 4.14** *An  $\omega$ -bounded  $n$ -manifold  $M$  with  $\chi(M) \neq 0$  has the ffp for flows.*

**Conjecture 4.15**  *$\omega$ -bounded  $n$ -manifolds have finite-dimensional (singular) homology over the rationals, and are Lefschetz spaces, i.e., any self-map with non-zero Lefschetz number has a fixed point.*

#### 4.4 *Ad hoc* hairy balls via Cantorian rigidity

Theorem 4.5 does not apply to  $\mathbb{L}_+^2 = \mathbb{L}_+ \times \mathbb{L}_+$  the square of the open long ray  $\mathbb{L}_+$ , nor does Proposition 4.8 apply to  $\mathbb{L}^2 \# \mathbb{L}^2$  or to the long plane surmounted by a cylindrical tower  $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$ , where  $\mathbb{L}_{\geq 0}$  is the closed long ray. Nonetheless, it turns out that all these surfaces lack a brush. Basically, the reason is that on the planar pipe the underlying foliation is—by the “Cantor rigidity” of [4]—forced to behave asymptotically like concentric squares, yielding a global long cross-section whose return times falsify the global topology of the planar pipe to a cylindrical one. (Recall that a continuous real-valued function on the long ray is eventually constant.) Thus, the ultimate reason is again the form of Cantor rigidity effecting that a cylindrical pipe differs radically from a planar pipe. We detail this *ad hoc* rigidity argument in the first situation (the other cases are similar):

**Proposition 4.16** *The long quadrant  $Q = \mathbb{L}_+^2$  does not support a brush.*

**Proof.** We divide it in two steps:

STEP 1: A SPECIAL CASE. *The foliation  $\mathcal{F}$  of  $Q$  by short lines bifurcating when they cross the diagonal of  $Q$  (with leaves of the form  $L_\alpha = (\{\alpha\} \times ]0, \alpha]) \cup ([0, \alpha] \times \{\alpha\})$ ,  $\alpha \in \mathbb{L}_+$ ) does not come from a flow.*

By contradiction, assume the existence of a flow  $f$  whose phase-portrait induces  $\mathcal{F}$ . For each  $\alpha \geq 1$ , we measure the time  $\tau(\alpha)$  elapsed until the point  $(\alpha, 1)$  reaches the position  $(1, \alpha)$ . (This is the time needed to travel from the horizontal cross-section  $\Sigma_h = \mathbb{L}_{\geq 1} \times \{1\}$  to the vertical one  $\Sigma_v = \{1\} \times \mathbb{L}_{\geq 1}$ ; after a time-reversion we assume the motion to have the “right” orientation.) The function  $\tau$  is real-valued, continuous and defined on  $\mathbb{L}_{\geq 1}$ , so must be eventually constant. This would imply that  $\mathbb{L}_{\geq 1} \times \mathbb{L}_{\geq 1}$  is homeomorphic to a long strip  $S = \mathbb{L}_{\geq 0} \times [0, 1]$ . This is however a falsification of the global topology: indeed it is easy to show that these surfaces are not homeomorphic, e.g. they can be distinguished by looking at foliations on their doubles. [Recall from [4] that the double  $2S$ , which is  $\Lambda_{0,1} := (\mathbb{S}^1 \times \mathbb{L}_{\geq 0}) \cup \mathbb{B}^2$ , has no foliation.]

STEP 2: THE GENERAL CASE. We shall reach a reduction to Step 1. Let  $f$  be a brush on  $Q$ , and look at the induced foliation (Theorem 2.2). Cut  $Q$  along the diagonal, and apply [4] to conclude that each octant  $O_i$  ( $i = 1, 2$ ) is (asymptotically) foliated by straight long rays or by straight short segments. The first option is to be excluded for a flow, hence both octants  $O_i$  are foliated by short segments occurring for a club  $C_i \subset \mathbb{L}_+$ . Passing to their common intersection  $C = C_1 \cap C_2$ , we find a new club where all the segments piece nicely together along the diagonal. A resemblance with case of Step 1 is emerging, except for a lack of complete “straightness” of the



foliation. Cutting the foliation along the broken lines  $L_\alpha$  for  $\alpha \in C$ , we get subregions which are strips  $[0, 1] \times \mathbb{R}$ . Typically we may meet a Reeb foliation, trouble-shooting the well-definition of the map  $\tau$ : orbits starting from the horizontal cross-section  $\Sigma_h$  instead of reaching the vertical one  $\Sigma_v$ , will loop-back like boomerangs. The issue is that Reeb components cannot be too numerous, for an infinitude of them implies a clustering onto a singularity front, imposed by a visual compression under the Cantor perspective (use the fact that an increasing  $\omega$ -sequence in  $\mathbb{L}_+$  is convergent). Jumping over these finitely many Reeb components by looking sufficiently far away (say  $\alpha \geq \alpha_0$ ) we may define the function  $\tau: \mathbb{L}_{\geq \alpha_0} \rightarrow \mathbb{R}$  to conclude as in Step 1. ■

#### 4.5 Abstract separation yoga via the five lemma

We shall now redirect our attention to the pseudo-Moore problem (Question 3.13) of showing that a surface sharing *in abstracto* the distinctive features of the Moore surface lacks a brush. To prepare the terrain, this section trains some abstract non-sense “Jordan” separation yoga. Below, singular homology is understood and coefficients are in  $\mathbb{Z}$ .

**Lemma 4.17** *Let  $J$  be closed set of a space  $M$ . Assume  $J \subset U$  strictly contained in an open set  $U$  of  $M$ .*

- (i) *If  $J$  divides  $M$ , then  $J$  divides  $U$ .*
- (ii) *If  $J$  divides  $U$  and if the map  $H_1(U) \rightarrow H_1(M)$  is surjective, then  $J$  divides  $M$ .*

**Proof.** Superpose the two exact sequences of the pairs  $(U, U - J)$  and  $(M, M - J)$ :

$$\begin{array}{ccccccccc} H_1(U) & \rightarrow & H_1(U, U - J) & \rightarrow & H_0(U - J) & \rightarrow & H_0(U) & \rightarrow & H_0(U, U - J) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_1(M) & \rightarrow & H_1(M, M - J) & \rightarrow & H_0(M - J) & \rightarrow & H_0(M) & \rightarrow & H_0(M, M - J) = 0 \end{array}$$

The second down-arrow is isomorphic by excision. Recall the *five lemma*: suppose that the diagram of abelian groups has exact rows and each square is commutative:

$$\begin{array}{ccccccccc} C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \rightarrow & C_4 & \rightarrow & C_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ D_1 & \rightarrow & D_2 & \rightarrow & D_3 & \rightarrow & D_4 & \rightarrow & D_5 \end{array}$$

Then

- (1) if  $f_2$  and  $f_4$  are epimorphisms and  $f_5$  is a monomorphism, then  $f_3$  is an epimorphism.
- (2) if  $f_2$  and  $f_4$  are monomorphisms and  $f_1$  is an epimorphism, then  $f_3$  is a monomorphism.

Conclude by observing that (1) proves (i), while (2) establishes (ii). ■

#### 4.6 Back to the pseudo-Moore question: separable hairy ball theorem

The following is another stagnation result, dual to Theorem 4.1, as it applies to separable manifolds (separability is the *désincarnation* of  $\omega$ -boundedness as the conjunction of both point-set properties collapses to compactness).

**Theorem 4.18** *Any flow on a simply-connected, separable, non-metric, non-bordered surface has a fixed point.*

**Proof.** By contradiction, let  $f: \mathbb{R} \times M \rightarrow M$  be a non-singular flow on such a surface  $M$ . By separability, let  $D$  be a countable dense subset of  $M$ . By phagocytosis, like any countable subset of a manifold,  $D$  can be engulfed inside a chart  $U$ , see [25, Prop.1]. Let  $\mathbb{R}U = f(\mathbb{R} \times U)$  be the orbit of that (dense) chart  $U$ . The set  $\mathbb{R}U$  is open, invariant (under the flow) and connected (as a continuous image of the set  $\mathbb{R} \times U \approx \mathbb{R}^3$ ). Moreover  $\mathbb{R}U$  is Lindelöf, hence cannot exhaust all of  $M$  (which is non-metric). Choose a point  $x \in M - \mathbb{R}U$ . A Poincaré-Bendixson argument shows the orbit  $\mathbb{R}x$  to be closed. Besides,  $\mathbb{R}x$  cannot be a circle, for a fixed point would be created by the non-metric Schoenflies [23], plus Brouwer. Hence  $\mathbb{R}x$  is a line (properly) embedded as closed set. Let  $T$  be a tubular neighbourhood of  $\mathbb{R}x$  (as usual constructed within a Lindelöf neighbourhood of the Lindelöf set  $\mathbb{R}x$  by metric combinatorial methods). This tube  $T$  has a bundle structure over a base which is contractible, so is trivial (Ehresmann-Feldbau-Steenrod)<sup>19</sup>. In particular  $\mathbb{R}x$  divides  $T$ , so by Lemma 4.17 (ii)  $\mathbb{R}x$  divides  $M$ . As  $\mathbb{R}U \subset M - \mathbb{R}x$ , the connectedness of  $\mathbb{R}U$  implies its containment in one component of  $M - \mathbb{R}x$ , but then  $\mathbb{R}U$  fails to be dense in  $M$ . This contradiction completes the proof. ■

Albeit quite general, the fixed-point theorems obtained so far (Theorems 4.5 and 4.18) have the serious limitation that they only apply to the simply-connected case. In the non-simply-connected case we may sometimes appeal to Proposition 4.8, at least if the long pipes have a nice canonical exhaustion, and dually appeal to Theorem 3.11 if there is a cytoplasmic decomposition. Let us however not miss the following:

<sup>19</sup>Here it is crucial that the base is metric (paracompact) as shown by the tangent bundle to the contractible classical version of the Prüfer surface ( $P$  collared). (Compare Spivak [67].)

**Corollary 4.19** *A separable non-metric surface (boundaryless) with countable fundamental group lacks a brush.*

**Proof.** Given a flow on such a surface, lift it to the universal covering (still separable) and apply (4.18). ■

## 5 Transitive flows

Beside non-singular flows (brushes), another weakening of minimality is *transitivity*:

**Definition 5.1** A flow on a topological space is *transitive* if it has at least one dense orbit. A space capable of a transitive flow is said to be *transitive*, and *intransitive* otherwise.

### 5.1 Transitivity obstructions (non-separability and dichotomy)

Obviously, a transitive flow implies separability of the phase-space (look at  $\mathbb{Q}x$  the rational times of a point with dense orbit). Thus,  $\mathbb{L}^2$  or the collared Prüfer surface,  $P_{\text{collar}}$ , are certainly intransitive. This does not inform us on the status of  $2P$ , the doubled Prüfer surface. Yet, for surfaces we have the classical Poincaré-Bendixson obstruction:

**Lemma 5.2** *A dichotomic surface (i.e., divided by any embedded circle) is intransitive.*

**Proof.** (Again the Bendixson sack argument.) By contradiction, let  $x \in M$  have a dense orbit under the flow  $f$ . Draw a cross-section  $\Sigma_x$  through  $x$  and consider an associated flow-box  $f([-\varepsilon, \varepsilon] \times \Sigma_x)$ . The point  $x$  must eventually return to  $\Sigma_x$ , so the piece of trajectory from  $x$  to its first return,  $x_1$ , closed up by the arc  $A$  of  $\Sigma_x$  joining  $x$  to  $x_1$ , defines a circle  $J$  in  $M$ . The component of  $M - J$  containing the near future of  $x_1$  (e.g.  $f(\varepsilon/2, x_1)$ ) will contain the full future of  $x_1$ . Then the “short” past of the arc  $A$  namely  $f([- \varepsilon, 0] \times \text{int} A)$  is an open “subrectangle” of the flow-box which the orbit of  $x$  will never revisit again. A contradiction. ■

This shows the intransitivity of  $2P$  which is clearly dichotomic (see the next section for a detailed argument). In particular, simply-connected surfaces are dichotomic (by the non-metric Jordan curve theorem in [23]<sup>20</sup>), hence intransitive. This holds for the Moore surface and the *Maungakiekie* (i.e., the result of a long cytoplasmic expansion of the 2-cell, as discussed in (3.3)), etc.

### 5.2 Dichotomy: heredity and Lindelöf approximation

This section collects some basic lemmas on *dichotomy* (i.e., the Jordan curve theorem holds true globally), establishing in particular the dichotomy of the doubled Prüfer surface,  $2P$ , i.e. Calabi-Rosenlicht’s version [15]. The reader with a good visual acuity can safely skip this section without loss of continuity.

**Lemma 5.3** *Any open set of a dichotomic surface is itself dichotomic.*

**Proof.** This follows at once from Lemma 4.17 (i). ■

The converse—in order to be non-tautological—takes the following form:

**Lemma 5.4** *A surface each of whose Lindelöf subsurfaces are dichotomic is itself dichotomic.*

**Proof.**<sup>21</sup> This argument has some variants depending on the amount of geometric topology inferred [we bracket the more geometric variants]. Assume by contradiction the existence of a Jordan curve  $J \subset M$  in the surface,  $M$ , such that  $M - J$  is connected. Choose  $U$  a Lindelöf subsurface with  $U \supset J$ . [We could take for  $U$  a tube around  $J$ .] The set  $U - J$  has at most countably many components  $U_i$ ,  $i = 1, 2, \dots$  [If  $U$  is a tube then exactly two components.] Pick a point  $x_i \in U_i$  in each component. Consider the countable set  $C = \{x_1, x_2, \dots\}$  in the (connected) manifold  $M - J$ . By arc-wise connectivity, there is pathes  $c_i$  from  $x_1$  to  $x_i$ . Covering by charts the union of those pathes,  $C$  can be engulfed in a connected Lindelöf open set  $L \subset M - J$ . [One could take for  $L$  a chart (taking advantage of the phagocytosis lemma).] The set  $U \cup L$  is open and Lindelöf. We have  $(U \cup L) - J = (U - J) \cup L = \bigcup_{i \in \mathbb{N}} U_i \cup L$ . Regarding this union as  $L$  plus the sets  $U_i$  (each meeting  $L$ ), it follows (by general topology) that this “bouquet-like” union is connected. This contradicts our assumption of “Lindelöf dichotomy”. [In the variant where  $U$  is a tube, we also find  $L \subset M - J$  a Lindelöf connected surface containing  $\{x_1, x_2\}$ : cover by charts (of  $M - J$ ) the path  $c_2 \subset M - J$  (joining  $x_1$  to  $x_2$ ) while keeping only the component of this union of charts which contains  $c_2$ . Again  $(U \cup L) - J = (U - J) \cup L = U_1 \cup U_2 \cup L$ , which is connected, as the union of two connected sets with a common intersection.] ■

<sup>20</sup>Recall that the Hausdorff separation axiom is crucial, for it is easy to draw on the *branched plane* a circle which does not disconnect it. (The latter arises from 2 replicas of  $\mathbb{R}^2$  by gluing along an open half-plane.)

<sup>21</sup>Unfortunately one cannot take advantage of Lemma 4.17 (ii).

**Corollary 5.5** *The doubled Prüfer surface  $2P$  is dichotomic.*

**Proof.** By Lemma 5.4 it is enough to show that each Lindelöf subsurface  $L$  of  $2P$  is dichotomic. Consider the open covering of  $2P$  by the sets  $B_x$  consisting of both half-planes plus the pencil of rays through  $x \in \mathbb{R}$ . By Lindelöfness of  $L$ , one may extract a countable subcover  $B := \bigcup_{x \in C} B_x \supset L$ . It is plain that  $B$  embeds in the plane  $\mathbb{R}^2$ , and the dichotomy of  $L$  follows from Lemma 5.3 (plus the classical Jordan curve theorem). ■

**Remark 5.6** Such routine separation arguments are of course very easy, and overlap a remark by R. L. Moore, as reported by F. Burton Jones [38, p. 573, Sec. 4, Parag. 2], where it is observed that the Moore surface  $M$  is “globally” Jordan, i.e. dichotomic. (This can be checked along the same lines as what we did for  $2P$ .)

### 5.3 Surfaces with prescribed topology and dynamics

As we saw “simple” topology oft impedes “complicated” dynamics (e.g., dichotomy obstructs transitivity). This section works out the experimental side of the various topologico-dynamical interactions. As usual, examples are intended to test the exhaustiveness of the theoretical obstructions listed so far. Arguably, any knowledge of the world (resp. theory) starts and ends with experiments (resp. examples). An oblique hope is that a blend of topologico-dynamical prescriptions singles out subclasses in the jungle of (non-metric) surfaces, where a classification looks more tractable. This scenario will rarely happen, yet when, in the nihilist art-form of an empty-set *classifiant*. So we start by a selection of:

- *Topological attributes* including: metric, separable, simply-connected, dichotomic; *versus*,
- *Dynamical attributes* including: minimal, quasi-minimal<sup>22</sup>, non-singular, transitive.

Below, we have pictured a *Venn diagram* showing the mutual disposition of these subclasses inside the universe of all Hausdorff surfaces. We shall primarily ask for representatives in each subclasses, and secondarily for a classification if possible. Some accompanying comments on this diagram are in order:

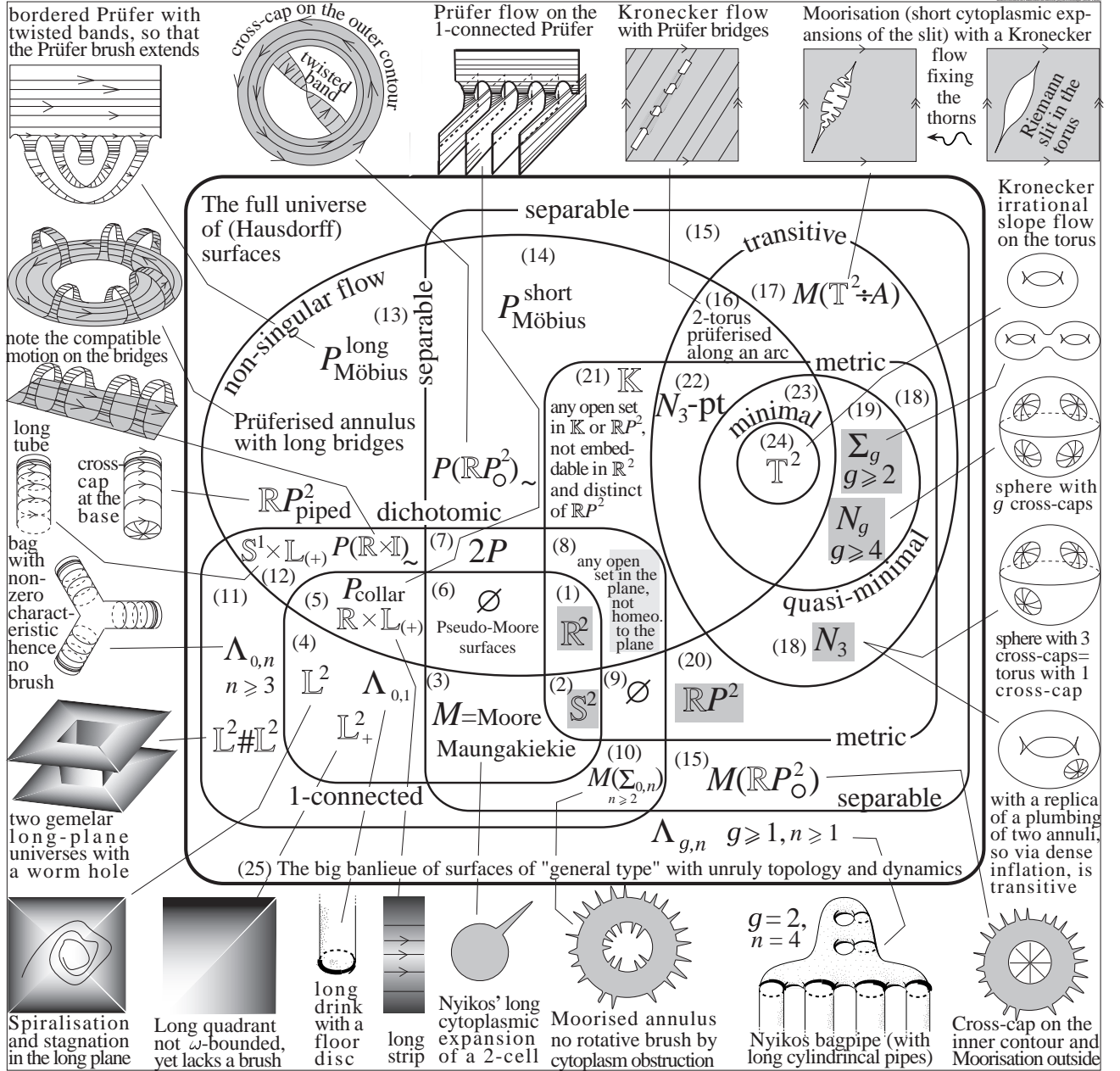
- (a) “Rounded rectangles” correspond to topological while “ovals” to dynamical attributes.
- (b) For identifying specimens the following symbolism is employed:  $P(\cdot)$  denotes the Prüferisation operator,  $M(\cdot)$  the Moorisation. (This is merely a matter of globalising the classic constructions of Prüfer and Moore, cf. eventually Definition 5.8.) Thus, the bordered Prüfer  $P$  is  $P(\mathbb{H})$  the Prüferisation of the upper half-plane  $\mathbb{H} = \mathbb{R} \times \mathbb{R}_{\geq 0}$ , while  $M$  the classic Moore surface is  $M(\mathbb{H})$ . A “circle index” means excising a 2-disc. A “tilde index” means that some identifications are made (usually on the boundary of a Prüferisation).
- (c) Notation:  $\mathbb{R}$  the real line,  $\mathbb{S}^1$  the circle,  $\mathbb{I} = [0, 1]$  the interval,  $\mathbb{L}_+ = ]0, \omega_1[$  and  $\mathbb{L}_{\geq 0} = [0, \omega_1[$  are the open (resp. closed) long rays,  $\mathbb{L}$  the long line,  $\mathbb{S}^2$  the 2-sphere,  $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$  the 2-torus,  $\Sigma_g$  the closed orientable surface of genus  $g$ ,  $\Sigma_{g,n}$  the same with  $n$  holes (2-discs excisions),  $N_g$  the closed non-orientable surface of genus  $g$  (defined in accordance with Riemann as the maximal number of disjoint non-dividing circles), thus  $N_g$  is the sphere with  $g$  cross-caps (hence  $\chi = 2 - g$ ). In particular,  $\mathbb{R}P^2 = N_1$  is the *projective plane*,  $\mathbb{K} = N_2$  is the *Klein bottle* (non-orientable closed surface with  $\chi = 0$ ). Finally  $\Lambda_{g,n}$  denotes the genus  $g$  surface with  $n$  pipes modelled on the cylinder  $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$ .
- (d) A shaded “manifold symbol” refers to the issue that the given manifold(s) turns out to be the *unique* representative(s) in the given class. The empty set symbol  $\emptyset$  indicates a class lacking any representative (we exclude the empty set to be a genuine surface).

The uniqueness of  $\mathbb{S}^2$  and  $\mathbb{R}^2$  in their respective classes follows from the classification of simply-connected metric surfaces. Recall the following key result (incarnating an advanced form of Poincaré-Bendixson theory beyond dichotomy):

**Lemma 5.7** *Among closed surfaces only  $\mathbb{S}^2, \mathbb{R}P^2$  and  $\mathbb{K}$  (Klein bottle) are intransitive.*

**Proof.** The intransitivity of  $\mathbb{S}^2$  and  $\mathbb{R}P^2$  follows from Poincaré-Bendixson (after lifting the flow to the universal covering in the second case). The intransitivity of  $\mathbb{K}$  was first established by Markley 1969 [45] (independently Aranson 1969), yet the argument of Gutiérrez 1978 [28, Thm 2, p. 314–315] seems to be intercontinentally recognised as the ultimate simplification. The transitivity of all remaining closed surfaces is observed in Peixoto 1962 [58, p. 113], also Blohin 1972 [11]. Beside the meticulous surgeries used by those authors, it is pleasant to recall the following cruder approach. Since a cross-cap diminishes  $\chi$  by one unit, we have the relation  $N_3 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \approx \mathbb{T}^2 \# \mathbb{R}P^2$ . Thus all other closed surfaces contain a replica of the plumbing of two open annuli (i.e. a punctured torus). Granting some geometric intuition, this open set can be inflated until to be dense. (Section 5.6 below discusses the issue that such dense inflations of open non-void sets might always be possible in separable manifolds.) By extending a Kronecker flow on this inflated punctured

<sup>22</sup>A flow is *quasi-minimal* if it has a finite number of fixed-points, while all non-stationary orbits are dense.



torus to the ambient closed surface one obtains the desired transitive flow. (This mechanism is further discussed below in Section 5.7.)

The uniqueness of  $\mathbb{R}P^2$  in its class is now easy. We seek after surfaces without brush but metric. Since open metric surfaces support brushes<sup>23</sup>, our surface must be compact. Intransitivity leaves *only* the three possibilities listed in Lemma 5.7. Non-dichotomy excludes  $\mathbb{S}^2$ , while  $\mathbb{K}$  is ruled out by the “no brush” condition, leaving  $\mathbb{R}P^2$  as the unique solution.

The emptiness of the class lying between  $\mathbb{S}^2$  and  $\mathbb{R}P^2$  (label (9) in our Venn diagram) is argued similarly. Such a surface must be compact (else it has a brush). Yet, the only closed dichotomic surface is  $\mathbb{S}^2$ .

We now embark in a more systematic exploration of our diagram by starting from the  $\mathbb{R}^2$  region, while spiraling clockwise (mimicking the numbering of parisian arrondissements):

(1) The first arrondissement is chosen as the one of  $\mathbb{R}^2$ , for the plane is not only locally but globally Euclidean. *The plane is characterised as the unique metric 1-connected surface bearing a brush.* Its only drawback is a certain dynamical poorness (transitivity is impeded by its dichotomy).

<sup>23</sup>In fact this result holds in any dimension, and even in the topological category (Lemma 5.21). In the 2-dimensional case the argument simplifies: introduce a smooth structure, find a Morse function without critical points (if any kill them by excising an arc starting from the singularity and running to infinity), and conclude by taking its gradient flow.

(2) Moving below we find  $\mathbb{S}^2$ , which is *the unique metric 1-connected surface lacking a brush*. Since the neighbouring class (9) is empty the statement can be sharpened into: *The 2-sphere is the unique dichotomic metric surface lacking a brush*.

(3) On the left of  $\mathbb{S}^2$ , we encounter the Moore surface  $M$  and the *Maungakiekie* (i.e., one long cytoplasmic expansion of the 2-cell, recall (3.3)). (Both lack a brush as they have a CTD, cf. Theorem 3.11.)

(4) Still more to the left we have  $\mathbb{L}^2$  and  $\Lambda_{0,1}$ . The long plane  $\mathbb{L}^2$  has no foliation of dimension 1 by short leaves, while  $\Lambda_{0,1}$  has no foliation at all [4], so both do not accept a brush. This follows also from Theorem 4.5 or Corollary 3.19. In view of Nyikos [55, p.669] this class contains a bewildering variety of specimens of cardinality  $2^{\aleph_1}$  of which the two above are just the most civilised examples. This class also includes non- $\omega$ -bounded examples, e.g.  $\mathbb{L}_+^2$  (Proposition 4.16).

(5) Moving up, we meet  $\mathbb{R} \times \mathbb{L}_{(+)}$  (the parenthetical “plus” means that we may take either the long line  $\mathbb{L}$  or the long ray  $\mathbb{L}_+$ ) and also  $P_{\text{collar}} := P \cup (\partial P \times \mathbb{R}_{\geq 0})$ , i.e. the original Prüfer surface which has a brush (3.5). (In view of Theorem 4.5, there is no  $\omega$ -bounded examples in this class.)

(6) This is an empty region corresponding to the pseudo-Moore problem (solved via Theorem 4.18.)

(7) Here we have  $2P$  the doubled Prüfer surface which has a brush (3.5) and is dichotomic (5.5). What else? (Try puncturing.)

(8) This class contains for instance the punctured plane  $\mathbb{R}^2 - \{0\}$  and more generally any open set of the plane (topologically distinct from the plane). This is a complete list of representatives due to the classification of (dichotomic) metric surfaces (compare e.g., Keréjártó [40]).

(9) This class is empty, as already argued.

(10) Here we have  $M(\Sigma_{0,n})$  for  $n \geq 2$  (recall that  $\Sigma_{g,n}$  denotes the compact orientable surface of genus  $g$  with  $n$  boundary components and that  $M$  is the Moorisation operation). For  $n = 2$  this surface is an annulus  $\Sigma_{0,2} = S^1 \times [0, 1]$  Moorised along its boundary. Since the genus  $g$  is zero these surfaces are dichotomic (apply Lemma 5.4), and they lack a brush (as they have a CTD). [Note that  $M(\Sigma_{0,1})$  belongs to arrondissement (3) being perhaps homeomorphic to the classic Moore surface  $M = M(\mathbb{H})$ , where  $\mathbb{H} = \mathbb{R} \times \mathbb{R}_{\geq 0}$ .]

(11) Now we have the surfaces  $\Lambda_{0,n}$  for  $n \geq 3$ . These surfaces having genus 0 are dichotomic (again Lemma 5.4), with non-trivial  $\pi_1$  (as soon as  $n \geq 2$ ) and finally lack a brush (either because in [4] it was shown that  $\Lambda_{g,n}$  has a foliation only when  $(g, n)$  is  $(1, 0)$  or  $(0, 2)$  or alternatively by Proposition 4.8). This class also contains the surface  $\mathbb{L}^2 \# \mathbb{L}^2$ , as discussed in Section 4.4.

(12) In this class we have  $\mathbb{S}^1 \times \mathbb{L}_{(+)}$ . What else? Again puncturing works. More sophisticated examples are obtainable by “rolling around” the tangent bundle of a smooth structure on  $\mathbb{L}_{(+)}$ . If  $L$  is a non-metric smooth 1-manifold, remove from its tangent bundle  $TL$  the zero-section to get two components  $TL^+$ ,  $TL^-$ . Scalar multiplication by a positive real  $\lambda \neq 1$  on  $TL^+$  yields a  $\mathbb{Z}$ -action on  $TL^+$ , whose quotient  $S := TL^+/\mathbb{Z}$  is a circle-bundle over  $L$ . We have then a brush  $f: \mathbb{R} \times S \rightarrow S$  induced by  $\phi: \mathbb{R} \times TL^+ \rightarrow TL^+$  given by  $\phi(t, v) = \lambda^t v$ , which induces an action of the circle  $\mathbb{R}/\mathbb{Z}$  on  $S$ . By Riemannian geometry the tangent bundle to a smooth non-metric manifold cannot be trivial [51], adumbrating that the surfaces  $S$  are not homeomorphic to the trivial circle bundle over  $L$ . Another example is the following: start with a strip  $\mathbb{R} \times [0, 1]$  and Prüferise its boundary, and then glue long bands to link boundary components with the same first coordinate. This surface, denoted  $P(\mathbb{R} \times \mathbb{I})_{\sim}$ , is not separable (because of the “longness” of the bands), is dichotomic (Lemma 5.4) (and incidentally with a  $\pi_1$  bigger than the previous ones).

(13) Here we give 3 examples: (1) start from  $P$  the bordered Prüfer surface  $P = (\mathbb{R} \times \mathbb{R}_{>0}) \sqcup \bigsqcup_{x \in \mathbb{R}} R_x$  endowed with its natural brush (windscreen wiper flow of Proposition 3.5). For each non-zero real  $x \in \mathbb{R} - \{0\}$ , we glue the boundary components  $R_x$  of  $P$  with the opposite  $R_{-x}$  in a way consistent with the flow. This produces many embedded copies of the Möbius band, so denote this surface by  $P_{\text{Möbius}}^{\text{short}}$ . To make the result non-separable we attach “long” Möbius bands (e.g. by first adding to  $P$  a closed collar  $\partial P \times [0, 1]$  and then performing the “flow compatible” identifications). Finally, aggregate an open collar to the “central” boundary component  $R_0$ . The resulting surface has a brush, is not separable and not dichotomic (it contains a Möbius band), hence non-orientable. (2) To get an orientable example Prüferise the annulus and link (radially) related components by “bridges” homeomorphic to  $\mathbb{R} \times [0, 1]$ . (3) Yet another example is the surface deduced from  $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$  by identifying via the antipodal map (cross-capping) the boundary. (This is  $\mathbb{R}P^2$  with a long pipe of the cylinder type.) This surface, call it  $\mathbb{R}P_{\text{piped}}^2$ , has a brush (in fact a circle action) and is not dichotomic (the image of the boundary in the quotient is a non-dividing circle). [From its description as  $\mathbb{R}P^2$  with a long (cylindrical) pipe, this surface can be shown to be *universally intransitive*, i.e. none of its open subset is transitive.]

(14) Here it is a bit more difficult to find examples. We will turn back to this after studying case (15). A natural candidate could be the surface  $P_{\text{Möbius}}^{\text{short}}$  constructed in the first part of (13); yet being separable, it is

difficult to ensure intransitivity. (However using the argument of (14bis) below, intransitivity will be clear.) [Another candidate could be a Prüferised annulus with short bridges, yet as this contains a replica of two plumbed annuli it is more likely that this example is transitive so belongs to (16).]

(15) Remove from  $\mathbb{R}P^2$  the interior of a closed 2-disc (to get a Möbius band) and Moorise the boundary to obtain  $M(\mathbb{R}P^2)$  (the “circle index” is a disc excision). This surface (call it  $S$ ) has a CTD, hence lacks a brush and is clearly separable. For belonging to class (15) it remains to prove intransitivity. By contradiction, let  $f: \mathbb{R} \times S \rightarrow S$  be a transitive flow. The surface  $S$  has a decomposition  $S = U \sqcup \bigsqcup_{x \in S^1} T_x$ , where  $U$  is an open Möbius band (=punctured  $\mathbb{R}P^2$ ), and each  $T_x$  is homeomorphic to  $\mathbb{R}_{\geq 0}$ . Let  $x \in S$  be a point with dense orbit  $\mathbb{R}x = f(\mathbb{R} \times \{x\})$ , and choose  $V$  a chart around  $x$ . Its orbit  $\mathbb{R}V := f(\mathbb{R} \times V) = \bigcup_{t \in \mathbb{R}} f_t(V)$  is open, Lindelöf and transitive under the restricted flow  $f: \mathbb{R} \times \mathbb{R}V \rightarrow \mathbb{R}V$ . The open cover of  $S$  by the sets  $U_x := U \cup T_x$  shows that  $\mathbb{R}V$  is contained in a countable union  $\bigcup_{x \in C} U_x$ , where  $C \subset S^1$  is countable. It is easy to show that  $\bigcup_{x \in C} U_x$  which is the core plus countably many thorns remains homeomorphic to the original core  $U$ , which is an (open) Möbius band (apply Morton Brown’s theorem). Now, the transitivity of  $\mathbb{R}V$  violates the universal intransitivity of  $\mathbb{R}P^2$ , i.e. all its open subsets are intransitive (cf. Lemma 5.12 below).

(14bis) [(14) revisited!] Separability makes hard to ensure intransitivity, yet we use the same trick as in (15) by taking advantage of the universal intransitivity of  $\mathbb{R}P^2$ . Thus, start with the projective plane visualised as a closed 2-disc modulo antipodes on the boundary. Remove the interior of a central disc to get  $W$  a surface with one boundary circle (a Möbius band). Consider the flow given by a rotational motion on this  $W$  (annulus with external circle identified by antipodes). Prüferise the intern circle, and consider a Prüfer flow. Then glue diametrically opposite boundaries in the way prescribed by the flow to obtain our surface  $S$ . (By construction it has a brush and is separable.) To check intransitivity, we argue by contradiction as before. Choose a chart  $V$  around a point  $x$  with a dense orbit, and note that  $f(\mathbb{R} \times V)$  is *a fortiori* transitive and Lindelöf, hence contained in the “core”  $\text{int}(W)$  plus countably many “bridges”. Denote by  $S_\omega$  this “countable approximation” of  $S$ . Attaching a single twisted band amounts to a single puncturing (if there were no twist this would produce *two* punctures!). [This can also be checked either by cut-and-past or via the classification of compact surfaces, after aggregating the natural boundary.] Arguing inductively  $S_\omega$  is in fact homeomorphic to a  $\omega$  times punctured Möbius band; against the universal intransitivity of  $\mathbb{R}P^2$ .

(16) A simple example is the 2-torus Prüferised along an arc. This surface has a transitive brush deduced from a windscreen wiper motion (as suggested in Example 3.6). Yet, we promised a formal treatment of (3.6) and this relies on Lemma 5.9 below. We apply the latter to  $W$  the result of a *Riemann slit* along a piece of orbit of a Kronecker flow (*slitting* merely amounts to duplicate each interior point of the segment). We take care of removing the two extremities of the “slit”. Thus,  $W$  is non-compact and with *two* boundary-components (“lips”). Equip  $W$  with the Kronecker flow suitably slowed down by multiplying its velocity vector field by a smooth positive function vanishing precisely on the two lips. Further, arrange a linear decay (of speed) when approaching the “lips”, then case (1) of Lemma 5.9 gives the required flow (after piecing together the boundaries of the Prüferisation  $P(W)$  lying “opposite”, i.e., those which were indexed by the same point prior to the slit).

(17) Again we just show an example: start with the torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  with an irrational flow  $f$ . Take a portion of trajectory  $A = f([t_1, t_2] \times \{x\})$  (say contained in the fundamental domain). Slit (à la Riemann) the torus along this arc  $A$  to get a bordered surface  $W = \mathbb{T}^2 \div A$  (imagine again that points of the interior of the arc are duplicated). Then Prüferise  $W$  to get  $P(W)$  and finally Moorise to get the surface  $M(W)$  which lacks a brush because it has a CTD. It remains to find a transitive flow. This involves the idea that if one alters the Prüfer flow (which has a linear speed decay when approaching the boundary), into one having a quadratic speed decay one obtains a quadratic Prüfer flow fixing point-wise the boundary of  $P$  and so descends on the Moore surface by fixing the “thorns” (in the classic case an explicit formula is  $f(t, (x, y)) = (x + ty^2, y)$ ). In view of the differential geometric character of the Prüfer construction (think with “rays”), this construction clearly globalises. The following two items should throw more light on this aspect:

**Definition 5.8** (*Prüferisation–Moorisation*) Given a bordered metric surface  $W$  (with a smooth structure and a Riemannian metric). One defines its *Prüferisation*  $P(W) = \text{int}W \sqcup \bigsqcup_{x \in \partial W} R_x$  by aggregating to the interior of  $W$ , all the “interior” rays in the tangent 2-planes,  $T_x W$ , with  $x \in \partial W$ . A topology on  $P(W)$  is introduced by mimicking the Prüfer topology, making  $P(W)$  into a bordered non-metric surface whose boundary components are the sets  $R_x$  each homeomorphic to  $\mathbb{R}$ . The *Moorisation*  $M(W)$  of  $W$  refers to the (boundaryless) surface, quotient of  $P(W)$  by self-gluing each of its boundary-components  $R_x$  via the involution given by reflecting “rays” about the ray at  $x$  orthogonal to the boundary.

**Lemma 5.9** (*Generalised Prüfer flows*) Given a smooth flow  $f$  on  $W$  fixing point-wise  $\partial W$ , there is a canonically induced “Prüfer flow”, denoted  $P(f)$ , on the Prüferisation  $P(W)$ . Two special cases are of interest:—(1) if the flow  $f$  has a linear speed decay when approaching the boundary then geodesic-rays undergo a “windscreen wiper motion” via  $f$ , while—(2) if this decay is quadratic then geodesic-rays deform into parabolas keeping the

same tangent. Thus, in case (1) the  $P(f)$  has no fixed point (on  $\partial P(W)$ ), while in case (2) all points of  $\partial P(W)$  are fixed under  $P(f)$ . Case (1) corresponds infinitesimally to (3.5), while case (2) induces a flow  $M(f)$  on the Moorisisation  $M(W)$  fixing the “thorns” point-wise.

**Proof.** We merely define the Prüfer flow  $P(f)$ . For a point in the interior of  $P(W)$ , just let act the flow  $f$ . If instead the point is a “ray”, choose a tangent vector, represent it by a path-germ, and let it evolve in time with the flow on  $W$  (until the given time  $t$  is elapsed), and take its tangent vector to define the image ray. ■

(18) Since open metric implies a brush, any example in this class must be compact. For  $g \geq 4$ ,  $N_g$  (closed non-orientable surface of genus  $g$  with  $\chi = 2 - g$ ) admits a *quasi-minimal flow*<sup>24</sup> (see Gutiérrez [29, Prop. 1]); and so belongs to class (19). In contrast for  $g = 3$ , it is known that  $N_3$  has no quasi-minimal flow (cf. the discussion in Section 5.5 below); so belongs to class (18). In view of the geographical location of all other closed surfaces (cf. (19) for  $\Sigma_g$ ,  $g \geq 2$ ), it turns out that  $N_3$  is the unique representant in this class.

(19) Here we have the closed orientable surfaces  $\Sigma_g$  of genus  $g \geq 2$  (with  $\chi < 0$ ). A quasi-minimal flow is obtained, by expressing  $\Sigma_g$  as a two-sheeted branched covering of the 2-torus  $\pi: \Sigma_g \rightarrow \mathbb{T}^2$ , and then lifting an irrational flow. This flow on  $\Sigma_g$  has only dense orbits, except those corresponding to the  $2(g-1)$  ramification points of the map  $\pi$  which are saddle points. As discussed in (18), the class (19) contains also the surfaces  $N_g$  with  $g \geq 4$ , yet not a single open surface. Hence the class (19) is also completely classified.

(20) As already discussed this class contains only the projective plane  $\mathbb{R}P^2$ .

(21) Here we meet the Klein bottle  $\mathbb{K}$  (intransitive by Lemma 5.7). In the compact case it is the only example, for the only closed surfaces with a brush (hence  $\chi = 0$ ) are  $\mathbb{T}^2$  and  $\mathbb{K}$ , but the former is transitive. What about non-compact examples? Using Lemma 5.12 below, one can certainly take the punctured Klein bottle. Granting the *inflation principle*, any open set in  $\mathbb{K}$  or  $\mathbb{R}P^2$  is intransitive (otherwise inflate the set to be dense keeping its homeomorphism type unchanged, and extend the transitive flow to the ambient closed surface). Thus, all those open sets belong to this class provided not embeddable in  $\mathbb{R}^2$  nor equal to  $\mathbb{R}P^2$ .

(22) Such an example if it exists must be open. Further by Benière’s result (Theorem 5.10 below) it must be non-orientable. Since  $N_3$  lacks a quasi-minimal flow (Section 5.5),  $N_3$  punctured once cannot be quasi-minimal; so  $N_3 - \text{pt}$  belongs to (22). (Of course the same applies to  $N_3$  minus a finite set.)

(23) As before, an example if it exists must be open and non-orientable. A candidate is  $N_4$  punctured once.

(24) Here we have the ideal “minimal” dynamics. In the compact case we have only  $\mathbb{T}^2$  (for H. Kneser 1924 [42] shows that any foliation on  $\mathbb{K}$  has a compact (hence circle) leaf). Puncturing the torus one finds many non-compact examples. The following result of Benière [8] provides much more:

**Theorem 5.10** *An open metric surface which is orientable, yet not embeddable in  $\mathbb{S}^2$  has a minimal flow.*

Lemma 5.12 below implies that the orientability assumption cannot be relaxed in Benière’s result (consider a punctured  $\mathbb{R}P^2$ ). (Caution: Benière’s theorem is sometimes quoted without the orientability proviso; compare Nikolaev-Zhuzhoma [54, p. xi, p. 252].) Nevertheless, a non-orientable surface may well admit a minimal flow, as shown by Gutiérrez’s construction [29] of a quasi-minimal flow on  $N_4$  (closed non-orientable surface of genus 4) with two hyperbolic saddles as unique singularities (other non-stationary orbits are dense). Consequently  $N_4 - \{2 \text{ pts}\}$  (two punctures) has a minimal flow. (Thus, non-orientable manifolds may well support minimal flows, answering partially a question of Gottschalk [27]. To get a compact example one must in view of Kneser [42] move to dimension 3, where one can suspend a minimal homeomorphism of the Klein bottle constructed by Ellis.) A natural problem would be a complete classification of surfaces with a minimal flow (specialists are probably quite close to the goal?).

(25) This is the big remaining banlieue: we merely mention  $\Lambda_{g,n}$  for  $g \geq 1, n \geq 1$ .

## 5.4 Transitivity transfers (up and down)

This section exposes two basic results relevant to our previous discussion (Section 5.3). (We present arguments in term of vector fields using the smoothing theory of Gutiérrez, yet one could also work with  $C^0$ -flows using Beck’s technique, briefly discussed in Section 5.8.)

**Lemma 5.11 (Transitivity preservation under finite puncturing).** *Let  $\Sigma$  be any closed transitive surface, then  $\Sigma$  punctured by a finite set  $F$  is also transitive.*

<sup>24</sup>i.e., all orbits are dense, except for a finite number of stationary points. (We follow the terminology of Gutiérrez-Pires [31] (*supertransitive* or *highly transitive* are used in the same or related contexts by other authors).

**Proof.** We may assume the flow smooth by Gutiérrez 1986 [30]. (On examples smoothness is satisfied.) Consider  $\xi$  the corresponding velocity vector field. Let  $x \in \Sigma$  have a dense orbit. We may perform the punctures outside this orbit. Take  $\varphi$  a non-negative smooth function on  $\Sigma$  vanishing exactly on  $F$ . Integrating the vector field  $\varphi\xi$  yields a flow on  $\Sigma - F$  whose orbit of  $x$  remains dense, indeed identic to the original trajectory. ■

Here is a reverse engineering:

**Lemma 5.12 (Intransitivity preservation under closed excision).** *Assume the closed surface  $\Sigma$  intransitive, then  $\Sigma - F$  ( $F$  being an arbitrary closed (meagre) subset of  $\Sigma$ ) is intransitive as well. Granting the inflation conjecture (5.15), any surface which embeds in  $\mathbb{S}^2$ ,  $\mathbb{RP}^2$  or  $\mathbb{K}$  is intransitive.*

**Proof.** Assume  $\Sigma - F$  transitive under the flow  $f$  which we assume smooth. (This involves the open case of Gutiérrez’s smoothing theory, alternatively use Beck’s technique.) Let  $x \in \Sigma - F$  be a point with dense orbit and let  $\xi$  be the velocity vector field of the flow  $f$ . Choose  $\varphi \geq 0$  a non-negative  $C^\infty$  function on  $\Sigma$  vanishing exactly on  $F$  (Whitney). Then the vector field  $\varphi\xi$  on  $\Sigma - F$  admits a smooth extension  $\eta$  to  $\Sigma$  vanishing on  $F$ . Integrating this field  $\eta$  (over the compact manifold  $\Sigma$ ) produces a flow  $f_\eta$  on  $\Sigma$  such that the orbit of  $x$  is dense in  $U = \Sigma - F$  (indeed identic to the original trajectory of  $x$ ), therefore dense in  $\Sigma$ . ■

Thus, finitely punctured projective planes and Klein bottles are still intransitive (hence belong to class (21) of the previous section). (Sharper conclusions are discussed in the next remark.) In particular  $\mathbb{K}$  is not the unique representant in its class. (Also this shows that orientability is essential to Benière’s result (5.10), e.g. the punctured projective plane (of genus 1) is intransitive.)

**Remark 5.13** We only proved Lemma 5.12 under the assumption that  $F$  is meagre (empty interior) or what is the same if its complement is dense. (For the application we made in (15) above, this weak form was sufficient as the set  $\mathbb{R}V = f(\mathbb{R} \times V)$  was dense.) Yet, to sharpen the method, it is desirable to dispose of the *inflation principle* (5.15), to the effect that any non-void open set of a (separable) manifold can be inflated to a dense subset while keeping its homeomorphism type intact. Then any open subset of these two surfaces  $\mathbb{RP}^2$  or  $\mathbb{K}$  is intransitive (hence belongs to class (21) provided it does not embed into the plane and is not all  $\mathbb{RP}^2$ ).

## 5.5 Non-quasiminimality of $N_3$ (Katok-Gutiérrez)

This section—slightly outside of our main theme—can be skipped without loosing continuity (its significance lies in completing our understanding of the metric-side of our Venn diagram in Section 5.3).

Our interest lies in the following proposition involving primarily authors like Katok-Blohin, Gutiérrez and Aranson-Zhuzhoma. The proofs in the literature are oft sketchy and in our opinion strangely cross-referenced. [For instance the statement in Nikolaev-Zhuzhoma [54, Lemma 7.4.1, p. 132] may contain a minor bug<sup>25</sup>. This and other sources (e.g. Aranson *et al.* [3]) observe that the assertion goes back to Katok, as reported in Blohin [11], where unfortunately no details are to be found.] Our argument lacks in rigor, yet we could not resist attempting a glimpse into the boosted Poincaré-Bendixson theory of the aforementioned authors. (Recall a flow is *quasi-minimal* if it has finitely many stationary points and all non-stationary orbits are dense.)

**Proposition 5.14** *The closed non-orientable surface of genus 3, denoted by  $N_3$ , has no quasi-minimal flow.*

**Proof.** By contradiction, assume  $N_3$  equipped with a quasi-minimal flow. The finitely many singular points all have a certain index. Positive indices (in the form of *sources* or *sinks*) are forbidden as they both imply a small circular cross-section enclosing the singular point, impeding transitivity. The case of a *center* is likewise excluded. Singularities of zero-indices (so-called *fake saddles*) are removable via a new flow *a fortiori* quasi-minimal. Then all singular points have negative indices. The Poincaré index formula imposes, as  $\chi(N_3) = -1$ , a unique singularity of index  $-1$  (a *hyperbolic saddle* with four separatrices).

A lemma of Peixoto-Gutiérrez [28, Lemma 2, p. 312] gives a global cross-section  $C$  to the flow. This circle  $C$  is *two-sided*, i.e. its tubular neighbourhood (being oriented by the flow lines) is an annulus (not a Möbius band). Moreover  $C$  is not dividing (a global separation would impede transitivity). Cutting  $N_3$  along the curve  $C$  yields a connected bordered surface  $W$  with two contours (boundary-components) with  $\chi$  unchanged equal to  $-1$ . Since the characteristic of a closed orientable surface is even ( $2 - 2g$  where  $g$  is the genus), and since two disc-excisions are required to create the two contours of  $W$  it follows from the oddness of  $\chi(W)$  that  $W$  is non-orientable. Hence  $W$  is an annulus with one cross-cap. The surface we started with,  $N_3$ , is recovered by gluing back the two contours. Naively two sewing seem possible, yet indistinguishable as  $W$  is non-orientable. For psychological convenience, we fix the radial identification (between the two contours of the annulus).

<sup>25</sup>Since the surface  $N_3$  is a torus with one cross-cap, one can start with a Kronecker flow on the torus, and deform it around the cross-cap, arranging the speeds to vanish on the “boundary” of the cross-cap. This gives a transitive flow on  $N_3$  with a circle of fixed points, which corrupts this Lemma 7.4.1. The latter seems therefore implicitly formulated under the finiteness assumption for the fixed-point set of the flow.



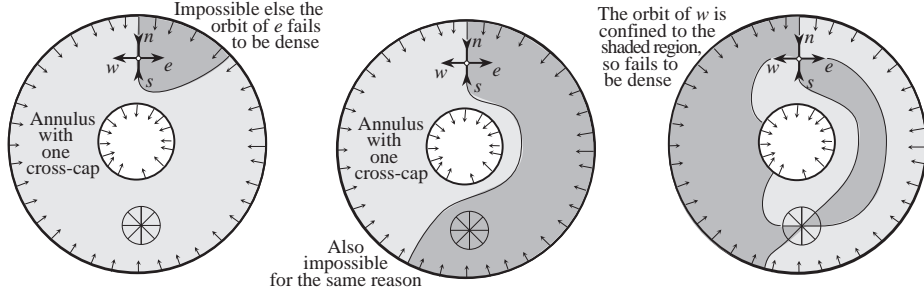


Figure 2: A heuristic Poincaré-Bendixson argument (à la Gutiérrez)

Figure 2 summarizes the situation: disjointly to the cross-cap is drawn the unique hyperbolic saddle. The flow is assumed entrant on the outer-boundary of  $W$  and sortant on the inner-boundary. Let the saddle be so oriented that its separatrices are directed in the four cardinal directions North-West-South-East, say with the North corresponding to an incoming (stable) separatrix converging to the singularity. Link the north-point  $n$  to the outer-boundary (such a crossing must actually occur since the orbit of  $n$  is dense). The south-point  $s$  cannot move directly to the outer-contour (Figure 2, left-side); this would impede either the east- or west-point to fill a dense orbit. Similarly  $s$  cannot reach the outer-contour as on the center-part of Figure 2, for in this case  $e$  is again trapped in the shaded sub-region. So  $s$  must travel through the cross-cap (Figure 2, right-side). Draw the forward-orbit of  $e$  until it reaches the inner-contour (while traversing the cross-cap), and extend also the forward-orbit of  $w$  until it intercepts the inner-contour. Then the orbit of the west-point  $w$  appears to be trapped in the shaded sub-region delimited by the four semi-orbits of the cardinal points (each extended until its first-passage through the cross-section  $C$ ), and so fails to be dense. This contradiction provides a vague completion of the proof. (A complete argument is certainly implicit in Gutiérrez 1978 [28].) ■

## 5.6 Densification-Inflation conjecture

The strong form of Lemma 5.12 (saying that each open set of an intransitive closed surface is itself intransitive) would be comforted by the following:

**Conjecture 5.15** (*Inflation conjecture*) *Assume that  $M$  is a connected separable manifold. Then for each non-empty open set  $U$  in  $M$ , there is an open set  $V$  dense in  $M$  and homeomorphic to  $U$ .*

The case where  $U$  is a chart holds true by virtue of the *phagocytosis lemma* in [25, Prop.1]. Indeed, as  $M$  is separable, one may select a countable dense subset  $D$  of  $M$ . Like any countable subset of a connected (Hausdorff) manifold,  $D$  is contained in a chart  $V$  (phagocytosis). This set  $V$  fulfils the desiderata of (5.15).

As expanded in the next section, an impetus for the *Inflation Conjecture* arises in the construction of transitive flows on manifolds. Yet, in view of the transitivity of the number-spaces  $\mathbb{R}^n$  ( $n \geq 3$ ), one can often bypass the inflation principle to apply instead phagocytosis. In case there should be a failure of (5.15), then it may hold in special circumstances (like DIFF, metric, compact, dimension two).

## 5.7 Construction of transitive flows: a recipe

A basic procedure to construct transitive flows on a (separable) manifold  $M$  is the following:

STEP 1. Find a *transistor* (or *transifold*)  $T$ , i.e. a manifold with a transitive flow (typically  $T$  will be a torus undergoing some puncture or the excision of some “small” closed set not jeopardizing its transitivity). Examples are given below.

STEP 2. Embed (whenever it is possible) the transistor  $T$  in the given manifold  $M$ .

STEP 3. Using the inflation principle (5.15)—alternatively some *ad hoc* construction—arrange the transistor  $T$  to be densely embedded in  $M$ . (Sometimes phagocytosis acts as a substitute.)

STEP 4. “Extend” the flow to the full manifold  $M$ . Then  $M$  will be the desired transitive manifold. The standard method uses vector fields (hence a smooth structure), but there is also a  $C^0$ -version (Beck’s technique) working at least when the manifold is metric (cf. Lemma 5.19). One hopes however that a non-metric version holds, if not in full generality at least in special circumstances (maybe when the manifold has nice functional properties).

In the surface-case (dimension 2), a “good” transistor is the punctured torus  $\mathbb{T}_*^2 = \mathbb{T}^2 - \{(0, 0)\}$ . Subdivide the fundamental domain  $[0, 1]^2$  in  $3^2 = 9$  subsquares. Puncturing (the origin) amounts to delete the 4 peripheral

subsquares, leaving a “Swiss cross” with opposite edges identified, i.e. the plumbing of two (open) annuli. This transistor embeds in many surfaces (e.g., in all closed surfaces distinct from  $\mathbb{S}^2$ ,  $\mathbb{R}P^2$  and  $\mathbb{K}$ ), and in fact in all metric surfaces distinct from the latter plus their open subsets. Besides, all strict open sets of  $\mathbb{S}^2$  and  $\mathbb{R}P^2$  embed in  $\mathbb{K}$ , for a punctured  $\mathbb{R}P^2$  is a Möbius band which embeds in  $\mathbb{K}$ . Thus, we recover the following classification of transitive metric surfaces (compare [37]):

**Proposition 5.16** *A metric connected surface is transitive if and only if it is not homeomorphic to  $\mathbb{S}^2$ ,  $\mathbb{R}P^2$  nor embeddable in the Klein bottle  $\mathbb{K}$ .*

**Proof.**  $\Rightarrow$  Otherwise, using an inflation (5.15) and an extension violates the intransitivity of  $\mathbb{K}$ .

$\Leftarrow$  Such a surface contains a copy of the transistor so apply the above “4 steps” recipe. ■

In dimension 3, a *universal* transistor is the 3-torus  $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  excised along the three circles axes  $T = \mathbb{T}^3 - (\mathbb{T}^1 \times \{0\} \times \{0\} \cup \{0\} \times \mathbb{T}^1 \times \{0\} \cup \{0\} \times \{0\} \times \mathbb{T}^1)$ . Universality of this transistor refers to its embedability in Euclidean space  $\mathbb{R}^3$  (hence in all 3-manifolds). [Indeed think of  $\mathbb{T}^3$  as the cube  $[0, 1]^3$  with opposite faces identified. Subdivide the segment  $[0, 1]$  in 3 subintervals, and accordingly the cube in  $3^3 = 27$  subcubes (*Rubik’s cube*). Deleting the 3 circles factor amounts to suppress all the “peripheral” subcubes of the Rubik’s cube (those with at least two visible faces) leaving  $1 + 6 = 7$  subcubes. Since opposite faces must be identified, we get (an open) cube with three handles, which embeds in  $\mathbb{R}^3$  without resistance.] Thus, the above recipe reproduces the following classic result; compare Oxtoby-Ulam 1941 [57] (compact polyhedrons of dimension  $\geq 3$ ), Sidorov 1968 [65] (transitivity of  $\mathbb{R}^n$ ,  $n \geq 3$ ), Anosov 1974 [2] (ergodicity of smooth compact manifolds  $M^n$ ,  $n \geq 3$ ):

**Proposition 5.17** *Any metric connected 3-manifold is transitive.*

Eventually, the ultimate generalisation could be:

**Conjecture 5.18** *Any separable connected 3-manifold (or of higher dimensions) is transitive.*

This sounds blatantly optimistic, yet the only difficulties appear to be located in the reparametrisation paradigm (i.e., Beck’s technique briefly discussed in the next section)—recall that in dimension  $\geq 3$ , the *inflation method* (5.15) is superseded by *phagocytosis* (since  $\mathbb{R}^3$  is a transistor, e.g. by an *ad hoc* inflation of the cube with 3 handles, or via Sidorov [65]).

## 5.8 Beck’s technique (plasticity of flows)

A basic desideratum, when dealing with flows, is a two-fold yoga of “restriction” and “extension”:

(1) *Given a flow on a space  $X$  and an open subset  $U \subset X$ , find a flow on  $U$  whose phase-portrait is the trace of the original one; and conversely:*

(2) *Given a flow on  $U$ , find a flow on  $X \supset U$  whose phase-portrait restricts to the given one.*

Thus, one expects that any open set of a brushing is itself a brushing, and that any separable super-space of a transitive space is likewise transitive, provided the sub-space is dense (or becomes so, after a suitable inflation).

Problem (1) is solved in Beck [6], when  $X$  is metric. (Example 5.20 below indicates a non-metric disruption.) The same technique of Beck (clever time-changes afforded by suitable integrations), solves Problem (2) in the metric case (compare [37, Lemma 2.3]):

**Lemma 5.19** *Let  $X$  be a locally compact metric space and  $U$  an open set of  $X$ . Given a flow  $f$  on  $U$ , there is a new flow  $f^*$  on  $X$  whose orbits in  $U$  are identic to the one under  $f$ .*

Tackling Conjecture 5.18 seems to involve an understanding of how much of the “extended” Beck technique (2) holds non-metrically. Even if the full swing of (2) should fail, there is certainly much room for partial results, say for separable 3-manifolds with a civilised geometry in the large (like 3D-avatars of the Prüfer or Moore manifolds, or perhaps those having sufficiently many functions, relating perhaps the question to *perfectly normality*).

**Example 5.20** Let  $X = \mathbb{R} \times \mathbb{L}_+$  be equipped with the natural (translation) flow  $f$  along the first real factor, and consider  $U = X - F$  the open set residual to  $F = \{0\} \times \omega_1$ . Then there is no flow  $f_*$  on  $U$  whose orbit-structure is the restriction of the one of  $f$  to  $U$ . [Using the *ad hoc* method of Section 4.4, it may be shown that the surface  $U = X - F$  is not a brushing.]

**Proof.** Assume by contradiction the existence of  $f_*$ . Chronometer the time  $\tau(x)$  required for a point  $(-1, x)$  starting from the cross-section  $\Sigma_{-1} := \{-1\} \times \mathbb{L}_+$  to reach  $\Sigma_1 := \{1\} \times \mathbb{L}_+$  under  $f_*$ . This defines a function  $\tau: \mathbb{L}_+ \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$  by letting  $\tau(x) = \infty$  if  $x \in \omega_1$ . Post-composing  $\tau$  with an arc-tangent function (extended by mapping  $\infty$  to  $\pi/2 = 1$ , for simplicity!) gives a continuous function  $\tau_*: \mathbb{L}_+ \rightarrow [0, 1]$ . Thus,  $\tau_* = 1$  on  $\omega_1$ , yet  $< 1$  outside, violating the fact that a real-valued continuous function on the long ray is eventually constant. ■

## 5.9 Morse-Thom brushes for metric open $C^0$ -manifolds, and Poincaré-Hopf

It is well-known, in the differentiable case at least, that an open metric manifold carries a critical-point free Morse function. (This reminds us the name of Thom, but are unable to recover our source!) At any rate, a proof is provided in M. W. Hirsch [34, p. 571], where rather the influence of Henry Whitehead is emphasized. In principle the result should extend to the topological case, to give:

**Proposition 5.21** *Any open metric  $C^0$ -manifold has a brush (where each orbit is a line).*

**Proof.** By results of Morse, Kirby-Siebenmann and Quinn—same references as in the proof of [4, Theorem 1.4]—it is known that there is a topological Morse function (the 4-dimensional case requires Quinn’s smoothing of open 4-manifolds [61]). One can alter the Morse function to have no critical points (via the usual trick of boring arcs, homeomorphic to  $[0, \infty)$ , escaping to infinity<sup>26</sup>). Like any submersion, this critical point free Morse function,  $f$ , defines a codimension-one foliation, to which we may apply Siebenmann’s transversality [66, Thm 6.26, p. 159] to obtain a dimension-one foliation transverse to the levels of  $f$ . The latter 1-foliation is clearly orientable (indeed oriented by increasing values of  $f$ ). By Whitney [71] there is a compatible flow for this foliation, which is the required brush. Conceivably, a  $C^0$ -theory of gradient flows may bypass foliations—thus, both Siebenmann and Whitney—yet going in the details will probably involve a common soup of technologies. (Of course, such “gradient” flows are dynamically very particular: each orbit of is a line (restrict  $f$  to the orbit), without “recurrences”, and with plenty of global cross-sections (any level-hypersurface of  $f$ .) ■

Let us briefly discuss—without the pretention of proving anything—a possible relevance of (5.21) to the  $C^0$ -avatar of *Heinz Hopf’s brushes* (i.e., the hypothetical existence of non-stationary flows on closed  $C^0$ -manifolds with vanishing Euler character,  $\chi = 0$ ). First, it is conceivable that to any  $C^0$ -flow with isolated singularities on a closed  $C^0$ -manifold one may—despite the lack of vector field interpretation—assign *indices* (also via the Brouwer degree); compare the procedure of Kerékjártó [41, p. 109] in the surface case, and also Dieudonné [20, p. 200]. Second, the *Poincaré-Hopf index formula* is likely to hold, i.e. indices add up to the characteristic of the manifold: like by Italian geometers, why not just trying to take advantage of the flow to push slightly the diagonal  $\Delta \subset M \times M$  into general position, to draw the index formula from the two-fold evaluation (algebraic vs. geometric) of the self-intersection number  $\Delta^2$ . [If not really convinced—owing to a lack of foundations—translate the geometric intuition into the cohomological language (of Moscow 1935: Alexander-Kolmogoroff-Whitney-Čech).] This would validate the index formula when  $M$  is orientable, and the general case follows by passing to the orientation covering. Third, given any closed  $C^0$ -manifold,  $M$ , puncture it once at a point,  $p \in M$ , to make it open and apply Proposition 5.21 to get a nonsingular flow on  $M - \{p\}$ . Using Beck’s technique (Lemma 5.19) the flow can be extended to  $M$  by fixing the point  $p$ . This would show that *any closed  $C^0$ -manifold has a “mono-singular” flow*, i.e., with a unique rest-point (well-known in the smooth case, [1]). Finally, in case  $\chi(M) = 0$ , one is tempted to claim that the unique singular point, having zero index, is removable. This would establish the desideratum.

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<sup>26</sup>Laurent Siebenmann calls them “ventilators”.

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Alexandre Gabard  
 Université de Genève  
 Section de Mathématiques  
 2-4 rue du Lièvre, CP 64  
 CH-1211 Genève 4  
 Switzerland  
 alexandregabard@hotmail.com

David Gauld  
 Department of Mathematics  
 The University of Auckland  
 Private Bag 92019  
 Auckland  
 New Zealand  
 d.gauld@auckland.ac.nz